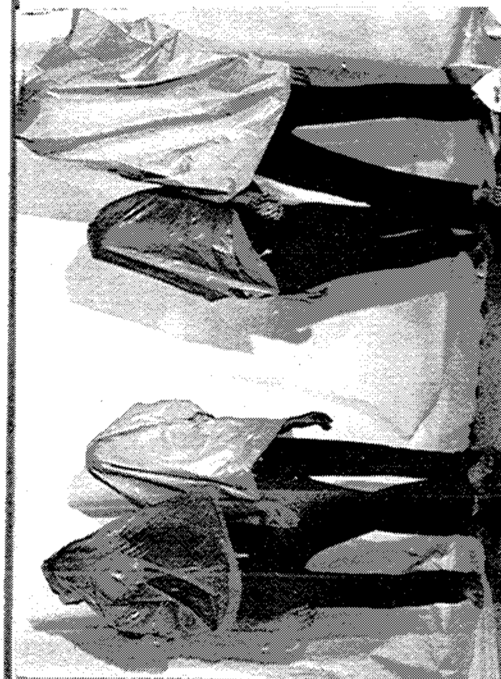
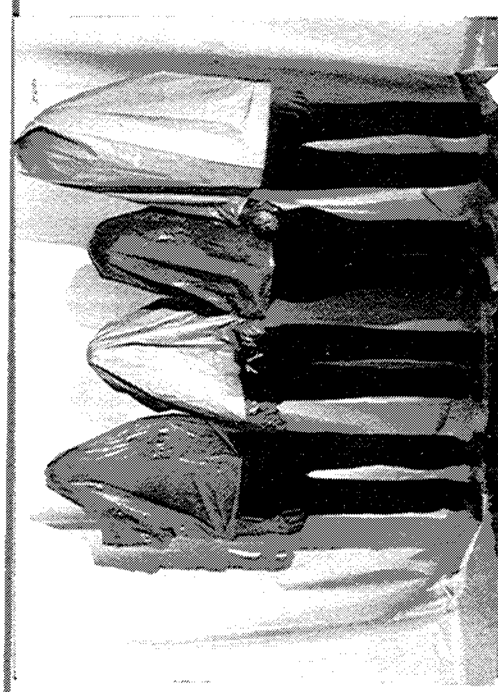
TABLE OF ENTROPIES H OF SIMPLE PARTITIONS

N = 4 = 2 + 2
H = 1.000000



N = 4 = 1 + 1 + 1 + 1
H = 2.000000

N = 4 = 4
H = 0.000000



N = 4 = 3 + 1
H = 0.811278

N = 4 = 2 + 1 + 1
H = 1.500000

N=4

3	1	0	0	0	0	0	0	0	0	0	0	.811278	2	2	0	0	0	0	0	0	0	0	0	0	1.000000
2	1	1										1.500000	1	1	1	1									2.000000

N=5

4	1	0	0	0	0	0	0	0	0	0	0	.721928	3	2	0	0	0	0	0	0	0	0	0	0	.970951
3	1	1										1.370951	2	2	1										1.521928

N=6

5	1	0	0	0	0	0	0	0	0	0	0	.650022	4	2	0	0	0	0	0	0	0	0	0	0	.918296
4	1	1	0	0	0	0	0	0	0	0	0	1.251629	3	3	0	0	0	0	0	0	0	0	0	0	1.000000
3	2	1	0	0	0	0	0	0	0	0	0	1.459148	2	2	2	0	0	0	0	0	0	0	0	0	1.584962
3	1	1	1									1.792481	2	2	1	1									1.91830

N=7

6	1	0	0	0	0	0	0	0	0	0	0	.591673	5	2	0	0	0	0	0	0	0	0	0	0	.863121
5	1	1	0	0	0	0	0	0	0	0	0	1.148835	4	3	0	0	0	0	0	0	0	0	0	0	.985228
4	2	1	0	0	0	0	0	0	0	0	0	1.378783	3	3	1	0	0	0	0	0	0	0	0	0	1.448816
3	2	2	0	0	0	0	0	0	0	0	0	1.556657	0	0	0	0	0	0	0	0	0	0	0	0	.000000

N=8

7	1	0	0	0	0	0	0	0	0	0	0	.543564	6	2	0	0	0	0	0	0	0	0	0	0	.811278
6	1	1	0	0	0	0	0	0	0	0	0	1.061278	5	3	0	0	0	0	0	0	0	0	0	0	.954434
5	2	1	0	0	0	0	0	0	0	0	0	1.298795	5	1	1	1	0	0	0	0	0	0	0	0	1.548795
4	4	0	0	0	0	0	0	0	0	0	0	1.000000	4	3	1	0	0	0	0	0	0	0	0	0	1.405639
4	2	2	0	0	0	0	0	0	0	0	0	1.500000	4	2	1	1	0	0	0	0	0	0	0	0	1.750000
3	3	2	0	0	0	0	0	0	0	0	0	1.561278	3	3	1	1	0	0	0	0	0	0	0	0	1.811278
3	2	2	1	0	0	0	0	0	0	0	0	1.905639	2	2	2	2	0	0	0	0	0	0	0	0	2.000000

N=9

8	1	0	0	0	0	0	0	0	0	0	0	.503258	7	2	0	0	0	0	0	0	0	0	0	0	.764204
7	1	1	0	0	0	0	0	0	0	0	0	.986427	6	3	0	0	0	0	0	0	0	0	0	0	.918296
6	2	1	0	0	0	0	0	0	0	0	0	1.224394	6	1	1	1	0	0	0	0	0	0	0	0	1.446617
5	4	0	0	0	0	0	0	0	0	0	0	.991076	5	3	1	0	0	0	0	0	0	0	0	0	1.351644
5	2	2	0	0	0	0	0	0	0	0	0	1.435520	5	2	1	1	0	0	0	0	0	0	0	0	1.657743
4	4	1	0	0	0	0	0	0	0	0	0	1.392147	4	3	2	0	0	0	0	0	0	0	0	0	1.530493
4	3	1	1	0	0	0	0	0	0	0	0	1.752715	4	2	2	1	0	0	0	0	0	0	0	0	1.836592
3	3	3	0	0	0	0	0	0	0	0	0	1.584962	3	3	2	1	0	0	0	0	0	0	0	0	1.891061
3	2	2	2	0	0	0	0	0	0	0	0	1.974937	0	0	0	0	0	0	0	0	0	0	0	0	.000000

N=10

9	1	0	0	0	0	0	0	0	0	0	0	.468996	8	2	0	0	0	0	0	0	0	0	0	0	.721928
8	1	1	0	0	0	0	0	0	0	0	0	.921928	7	3	0	0	0	0	0	0	0	0	0	0	.881291
7	2	1	0	0	0	0	0	0	0	0	0	1.156780	7	1	1	1	0	0	0	0	0	0	0	0	1.356780
6	4	0	0	0	0	0	0	0	0	0	0	.970951	6	3	1	0	0	0	0	0	0	0	0	0	1.295462
6	2	2	0	0	0	0	0	0	0	0	0	1.370951	6	2	1	1	0	0	0	0	0	0	0	0	1.570951
6	1	1	1	1	0	0	0	0	0	0	0	1.770951	5	5	0	0	0	0	0	0	0	0	0	0	1.000000
5	4	1	0	0	0	0	0	0	0	0	0	1.360964	5	3	2	0	0	0	0	0	0	0	0	0	1.485475
5	3	1	1	0	0	0	0	0	0	0	0	1.685475	5	2	2	1	0	0	0	0	0	0	0	0	1.760964
5	2	1	1	1	0	0	0	0	0	0	0	1.960964	4	4	2	0	0	0	0	0	0	0	0	0	1.521928
4	4	1	1	0	0	0	0	0	0	0	0	1.721928	4	3	3	0	0	0	0	0	0	0	0	0	1.570951
4	3	2	1	0	0	0	0	0	0	0	0	1.846439	4	3	1	1	1	0	0	0	0	0	0	0	2.046439

N=10

4	2	2	2	0	0	0	0	0	0	0	0	1.921928	4	2	2	1	1	0	0	0	0	0	0	0	2.121928
3	3	3	1	0	0	0	0	0	0	0	0	1.895462	3	3	2	2	0	0	0	0	0	0	0	0	1.970950
3	3	2	1	1	0	0	0	0	0	0	0	2.170951	3	2	2	2	1	0	0	0	0	0	0	0	2.246439
2	2	2	2	2	0	0	0	0	0	0	0	2.321928	0	0	0	0	0	0	0	0	0	0	0	0	.000000

N=11

10	1	0	0	0	0	0	0	0	0	0	0	.439497	9	2	0	0	0	0	0	0	0	0	0	0	.684038
9	1	1	0	0	0	0	0	0	0	0	0	.865857	8	3	0	0	0	0	0	0	0	0	0	0	.848351
8	2	1	0	0	0	0	0	0	0	0	0	1.095795	8	1	1	1	0	0	0	0	0	0	0	0	1.277613
7	4	0	0	0	0	0	0	0	0	0	0	.945660	7	3	1	0	0	0	0	0	0	0	0	0	1.240670
7	2	2	0	0	0	0	0	0	0	0	0	1.309297	7	2	1	1	0	0	0	0	0	0	0	0	1.491115
7	1	1	1	1	0	0	0	0	0	0	0	1.672933	6	5	0	0	0	0	0	0	0	0	0	0	.994030
6	4	1	0	0	0	0	0	0	0	0	0	1.322179	6	3	2	0	0	0	0	0	0	0	0	0	1.435371
6	3	1	1	0	0	0	0	0	0	0	0	1.617190	6	2	2	1	0	0	0	0	0	0	0	0	1.685816
6	2	1	1	1	0	0	0	0	0	0	0	1.867634	5	5	1	0	0	0	0	0	0	0	0	0	1.348588
5	4	2	0	0	0	0	0	0	0	0	0	1.494919	5	4	1	1	0	0	0	0	0	0	0	0	1.676737
5	3	3	0	0	0	0	0	0	0	0	0	1.539485	5	3	2	1	0	0	0	0	0	0	0	0	1.789929
5	3	1	1	1	0	0	0	0	0	0	0	1.971747	5	2	2	2	0	0	0	0	0	0	0	0	1.858555
5	2	2	1	1	0	0	0	0	0	0	0	2.040373	4	4	3	0	0	0	0	0	0	0	0	0	1.572624
4	4	2	1	0	0	0	0	0	0	0	0	1.823068	4	4	1	1	1	0	0	0	0	0	0	0	2.004886
4	3	3	1	0	0	0	0	0	0	0	0	1.867634	4	3	2	2	0	0	0	0	0	0	0	0	1.936260
4	3	2	1	1	0	0	0	0	0	0	0	2.118078	4	2	2	2	1	0	0	0	0	0	0	0	2.186704
3	3	3	2	0	0	0	0	0	0	0	0	1.980826	3	3	3	1	1	0	0	0	0	0	0	0	2.162644
3	3	2	2	1	0	0	0	0	0	0	0	2.231270	3	2	2	2	2	0	0	0	0	0	0	0	2.299896

N=12

11	1	0	0	0	0	0	0	0	0	0	0	.413817	10	2	0	0	0	0	0	0	0	0	0	0	.650022
10	1	1	0	0	0	0	0	0	0	0	0	.816689	9	3	0	0	0	0	0	0	0	0	0	0	.811278
9	2	1	0	0	0	0	0	0	0	0	0	1.040852	9	1	1	1	0	0	0	0	0	0	0	0	1.207519
8	4	0	0	0	0	0	0	0	0	0	0	.918296	8	3	1	0	0	0	0	0	0	0	0	0	1.188722
8	2	2	0	0	0	0	0	0	0	0	0	1.251629	8	2	1	1	0	0	0	0	0	0	0	0	1.418296
8	1	1	1	1	0	0	0	0	0	0	0	1.584962	7	5	0	0	0	0	0	0	0	0	0	0	.979869
7	4	1	0	0	0	0	0	0	0	0	0	1.280672	7	3	2	0	0	0	0	0	0	0	0	0	1.384431
7	3	1	1	0	0	0	0	0	0	0	0	1.551098	7	2	2	1	0	0	0	0	0	0	0	0	1.614005
7	2	1	1	1	0	0	0	0	0	0	0	1.780672	7	1	1	1	1	1	0	0	0	0	0	0	1.947339
6	6	0	0	0	0	0	0	0	0	0	0	1.000000	6	5	1	0	0	0	0	0	0	0	0	0	1.325011
6	4	2	0	0	0	0	0	0	0	0	0	1.459148	6	4	1	1	0	0	0	0	0	0	0	0	1.625815
6	3	3	0	0	0	0	0	0	0	0	0	1.500000	6	3	2	1	0	0	0	0	0	0	0	0	1.729574
6	3	1	1	1	0	0	0	0	0	0	0	1.896241	6	2	2	2	0	0	0	0	0	0	0	0	1.792481
6	2	2	1	1	0	0	0	0	0	0	0	1.959148	6	2	1	1	1	1	0	0	0	0	0	0	2.125815
5	5	2	0	0	0	0	0	0	0	0	0	1.483356	5	5	1	1	0	0	0	0	0	0	0	0	1.650022
5	4	3	0	0	0	0	0	0	0	0	0	1.554585	5	4	2	1	0	0	0	0	0	0	0	0	1.784159
5	4	1	1	1	0	0	0	0	0	0	0	1.950826	5	3	3	1	0	0	0	0	0	0	0	0	1.826011
5	3	2	2	0	0	0	0	0	0	0	0	1.887918	5	3	2	1	1	0	0	0	0	0	0	0	2.054585
5	3	1	1	1	1	0	0	0	0	0	0	2.221252	5	2	2	2	1	0	0	0	0	0	0	0	2.117492
5	2	2	1	1	1	0	0	0	0	0	0	2.284159	4	4	4	0	0	0	0	0	0	0	0	0	1.584962
4	4	3	1	0	0	0	0	0	0	0	0	1.855388	4	4	2	2	0	0	0	0	0	0	0	0	1.918296
4	4	2	1	1	0	0	0	0	0	0	0	2.084962	4	4	1	1	1	1	0	0	0	0	0	0	2.251629
4	3	3	2	0	0	0	0	0	0	0	0	1.959148	4	3	3	1	1	0	0	0	0	0	0	0	2.125814
4	3	2	2	1	0	0	0	0	0	0	0	2.188722	4	3	2	1	1	1	0	0	0	0	0	0	2.355388
4	2	2	2	2	0	0	0	0	0	0	0	2.251629	4	2	2	2	1	1	0	0	0	0	0	0	2.418296
3	3	3	3	0	0	0	0	0	0	0	0	2.000080	3	3	3	2	1	0	0	0	0	0	0	0	2.229574

N=12

3	3	3	1	1	1	0	0	0	0	C	0	2.396241	3	3	2	2	2	0	0	0	0	0	0	0	2.292481
3	3	2	2	1	1	0	0	0	0	C	0	2.459148	3	2	2	2	2	1	0	0	0	0	0	0	2.522055
2	2	2	2	2	2	0	0	0	0	C	0	2.584962	0	0	0	0	0	0	0	0	0	0	0	0	.000000

N=13

12	1	0	0	C	0	0	0	0	0	C	0	.391243	11	2	0	0	0	0	0	0	0	0	0	0	.619382
11	1	1	0	C	0	0	0	0	0	C	0	.773228	10	3	0	0	0	0	0	0	0	0	0	0	.779350
10	2	1	0	C	0	0	0	0	0	C	0	.991264	10	1	1	1	0	0	0	0	0	0	0	0	1.146110
9	4	0	0	0	0	0	0	0	0	C	0	.890492	9	3	1	0	0	0	0	0	0	0	0	0	1.140116
9	2	2	0	0	0	0	0	C	0	C	0	1.198184	9	2	1	1	0	0	0	0	0	0	0	0	1.352030
9	1	1	1	1	0	0	0	C	0	C	0	1.505876	8	5	0	0	0	0	0	0	0	0	0	0	.961237
8	4	1	0	0	0	0	0	0	0	C	0	1.238901	8	3	2	0	0	0	0	0	0	0	0	0	1.334679
8	3	1	1	0	0	0	0	0	0	C	0	1.488525	8	2	2	1	0	0	0	0	0	0	0	0	1.546593
8	2	1	1	1	0	0	0	0	0	C	0	1.700440	8	1	1	1	1	1	0	0	0	0	0	0	1.854286
7	6	0	0	0	0	0	0	C	0	C	0	.995727	7	5	1	0	0	0	0	0	0	0	0	0	1.296738
7	4	2	0	C	0	0	0	0	0	C	0	1.419556	7	4	1	1	0	0	0	0	0	0	0	0	1.573402
7	3	3	0	0	0	0	0	0	0	C	0	1.457266	7	3	2	1	0	0	0	0	0	0	0	0	1.669180
7	3	1	1	1	0	0	0	0	0	C	0	1.823026	7	2	2	2	0	0	0	0	0	0	0	0	1.727248
7	2	2	1	1	0	0	0	0	0	C	0	1.881095	7	2	1	1	1	1	0	0	0	0	0	0	2.034941
6	6	1	0	0	0	0	0	0	0	C	0	1.314320	6	5	2	0	0	0	0	0	0	0	0	0	1.460485
6	5	1	1	C	0	0	0	0	0	C	0	1.614331	6	4	3	0	0	0	0	0	0	0	0	0	1.526235
6	4	2	1	C	0	0	0	0	0	C	0	1.738149	6	4	1	1	1	0	0	0	0	0	0	0	1.891995
6	3	3	1	0	0	0	0	0	0	C	0	1.775859	6	3	2	2	0	0	0	0	0	0	0	0	1.833927
6	3	2	1	1	0	0	0	0	0	C	0	1.987773	6	3	1	1	1	1	0	0	0	0	0	0	2.141819
6	2	2	2	1	0	0	0	0	0	C	0	2.045841	6	2	2	1	1	1	0	0	0	0	0	0	2.199688
5	5	3	0	0	0	0	0	0	0	C	0	1.548581	5	5	2	1	0	0	0	0	0	0	0	0	1.760495
5	5	1	1	1	0	0	0	0	0	C	0	1.914341	5	4	4	0	0	0	0	0	0	0	0	0	1.576621
5	4	3	1	0	0	0	0	0	0	C	0	1.826245	5	4	2	2	0	0	0	0	0	0	0	0	1.884313
5	4	2	1	1	0	0	0	0	0	C	0	2.038160	5	4	1	1	1	1	0	0	0	0	0	0	2.192006
5	3	3	2	0	0	0	0	0	0	C	0	1.922023	5	3	3	1	1	0	0	0	0	0	0	0	2.076869
5	3	2	2	1	0	0	0	0	0	C	0	2.133937	5	3	2	1	1	1	0	0	0	0	0	0	2.287784
5	2	2	2	2	0	0	0	0	0	C	0	2.192006	5	2	2	2	1	1	0	0	0	0	0	0	2.346852
4	4	4	1	0	0	0	0	0	0	C	0	1.854286	4	4	3	2	0	0	0	0	0	0	0	0	1.950064
4	4	3	1	1	0	0	0	0	0	C	0	2.103910	4	4	2	2	1	0	0	0	0	0	0	0	2.161978
4	4	2	1	1	1	0	0	0	0	C	0	2.315824	4	3	3	3	0	0	0	0	0	0	0	0	1.987773
4	3	3	2	1	0	0	0	0	0	C	0	2.199688	4	3	3	1	1	1	0	0	0	0	0	0	2.353534
4	3	2	2	2	0	0	0	0	0	C	0	2.257756	4	3	2	2	1	1	0	0	0	0	0	0	2.411602
4	2	2	2	2	1	0	0	0	0	C	0	2.469670	3	3	3	3	1	0	0	0	0	0	0	0	2.237397
3	3	3	2	2	0	0	0	0	0	C	0	2.295466	3	3	3	2	1	1	0	0	0	0	0	0	2.449312
3	3	2	2	2	1	0	0	0	0	C	0	2.507380	3	2	2	2	2	2	0	0	0	0	0	0	2.565448

N=14

13	1	0	0	0	0	0	0	0	0	C	0	.371232	12	2	0	0	0	0	0	0	0	0	0	0	.591673
12	1	1	0	0	0	0	0	0	0	C	0	.734530	11	3	0	0	0	0	0	0	0	0	0	0	.749595
11	2	1	0	0	0	0	0	0	0	C	0	.946373	11	1	1	1	0	0	0	0	0	0	0	0	1.089230
10	4	0	0	0	0	0	0	0	0	C	0	.863121	10	3	1	0	0	0	0	0	0	0	0	0	1.094914
10	2	2	0	0	0	0	0	0	0	C	0	1.148835	10	2	1	1	0	0	0	0	0	0	0	0	1.291692
10	1	1	1	1	0	0	0	0	0	C	0	1.434549	9	5	0	0	0	0	0	0	0	0	0	0	.940286
9	4	1	0	0	0	0	0	0	0	C	0	1.198117	9	3	2	0	0	0	0	0	0	0	0	0	1.287054
9	3	1	1	0	0	0	0	0	0	0	0	1.429911	9	2	2	1	0	0	0	0	0	0	0	0	1.488832
9	2	1	1	1	0	0	0	0	0	0	0	1.626689	9	1	1	1	1	1	0	0	0	0	0	0	1.769546
8	6	0	0	0	0	0	0	0	0	C	0	.985228	8	5	1	0	0	0	0	0	0	0	0	0	1.263809

N=14

8	4	2	0	0	0	0	0	0	0	0	0	1.378783
8	3	3	0	0	0	0	0	0	0	0	0	1.413799
8	3	1	1	1	0	0	0	0	0	0	0	1.753434
8	2	2	1	1	0	0	0	0	0	0	0	1.807355
8	1	1	1	1	1	1	0	0	0	0	0	2.093069
7	6	1	0	0	0	0	0	0	0	0	0	1.295836
7	5	1	1	0	0	0	0	0	0	0	0	1.574417
7	4	2	1	0	0	0	0	0	0	0	0	1.689392
7	3	3	1	0	0	0	0	0	0	0	0	1.724408
7	3	2	1	1	0	0	0	0	0	0	0	1.921185
7	2	2	2	1	0	0	0	0	0	0	0	1.975106
7	2	1	1	1	1	1	0	0	0	0	0	2.260820
6	6	1	1	0	0	0	0	0	0	0	0	1.591673
6	5	2	1	0	0	0	0	0	0	0	0	1.727397
6	4	4	0	0	0	0	0	0	0	0	0	1.556657
6	4	2	2	0	0	0	0	0	0	0	0	1.842371
6	4	1	1	1	1	0	0	0	0	0	0	2.128085
6	3	3	1	1	0	0	0	0	0	0	0	2.020244
6	3	2	1	1	1	0	0	0	0	0	0	2.217022
6	2	2	2	2	0	0	0	0	0	0	0	2.128085
6	2	2	1	1	1	1	0	0	0	0	0	2.413799
5	5	3	1	0	0	0	0	0	0	0	0	1.809200
5	5	2	1	1	0	0	0	0	0	0	0	2.005978
5	4	4	1	0	0	0	0	0	0	0	0	1.835238
5	4	3	1	1	0	0	0	0	0	0	0	2.067031
5	4	2	1	1	1	0	0	0	0	0	0	2.263809
5	3	3	3	0	0	0	0	0	0	0	0	1.959190
5	3	3	1	1	1	0	0	0	0	0	0	2.298825
5	3	2	2	1	1	0	0	0	0	0	0	2.352746
5	2	2	2	2	1	0	0	0	0	0	0	2.406666
4	4	4	2	0	0	0	0	0	0	0	0	1.950212
4	4	3	3	0	0	0	0	0	0	0	0	1.985228
4	4	3	1	1	1	0	0	0	0	0	0	2.324863
4	4	2	2	1	1	0	0	0	0	0	0	2.378783
4	3	3	3	1	0	0	0	0	0	0	0	2.217022
4	3	3	2	1	1	0	0	0	0	0	0	2.413799
4	3	2	2	2	1	0	0	0	0	0	0	2.467720
4	2	2	2	2	2	0	0	0	0	0	0	2.521641
3	3	3	3	2	0	0	0	0	0	0	0	2.305958
3	3	3	2	2	1	0	0	0	0	0	0	2.502736
3	3	2	2	2	2	0	0	0	0	0	0	2.556657
3	2	2	2	2	2	1	0	0	0	0	0	2.753434
8	4	1	1	0	0	0	0	0	0	0	0	1.521641
8	3	2	1	0	0	0	0	0	0	0	0	1.610577
8	2	2	2	0	0	0	0	0	0	0	0	1.664498
8	2	1	1	1	1	0	0	0	0	0	0	1.950212
7	7	0	0	0	0	0	0	0	0	0	0	1.000000
7	5	2	0	0	0	0	0	0	0	0	0	1.431560
7	4	3	0	0	0	0	0	0	0	0	0	1.492614
7	4	1	1	1	0	0	0	0	0	0	0	1.832249
7	3	2	2	0	0	0	0	0	0	0	0	1.778328
7	3	1	1	1	1	0	0	0	0	0	0	2.064043
7	2	2	1	1	1	0	0	0	0	0	0	2.117963
6	6	2	0	0	0	0	0	0	0	0	0	1.448816
6	5	3	0	0	0	0	0	0	0	0	0	1.530619
6	5	1	1	1	0	0	0	0	0	0	0	1.870254
6	4	3	1	0	0	0	0	0	0	0	0	1.788450
6	4	2	1	1	0	0	0	0	0	0	0	1.985228
6	3	3	2	0	0	0	0	0	0	0	0	1.877387
6	3	2	2	1	0	0	0	0	0	0	0	2.074165
6	3	1	1	1	1	1	0	0	0	0	0	2.359879
6	2	2	2	1	1	0	0	0	0	0	0	2.270942
5	5	4	0	0	0	0	0	0	0	0	0	1.577406
5	5	2	2	0	0	0	0	0	0	0	0	1.863121
5	5	1	1	1	1	0	0	0	0	0	0	2.148835
5	4	3	2	0	0	0	0	0	0	0	0	1.924174
5	4	2	2	1	0	0	0	0	0	0	0	2.120952
5	4	1	1	1	1	1	0	0	0	0	0	2.406666
5	3	3	2	1	0	0	0	0	0	0	0	2.156968
5	3	2	2	2	0	0	0	0	0	0	0	2.209889
5	3	2	1	1	1	1	0	0	0	0	0	2.495603
5	2	2	2	1	1	1	0	0	0	0	0	2.549523
4	4	4	1	1	0	0	0	0	0	0	0	2.093069
4	4	3	2	1	0	0	0	0	0	0	0	2.182006
4	4	2	2	2	0	0	0	0	0	0	0	2.236926
4	4	2	1	1	1	1	0	0	0	0	0	2.521641
4	3	3	2	2	0	0	0	0	0	0	0	2.270942
4	3	3	1	1	1	1	0	0	0	0	0	2.556657
4	3	2	2	1	1	1	0	0	0	0	0	2.610577
4	2	2	2	2	1	1	0	0	0	0	0	2.664498
3	3	3	3	1	1	0	0	0	0	0	0	2.448816
3	3	3	2	1	1	1	0	0	0	0	0	2.645593
3	3	2	2	2	1	1	0	0	0	0	0	2.699514
2	2	2	2	2	2	2	0	0	0	0	0	2.807355

N=15

14	1	0	0	0	0	0	0	0	0	0	0	.353359
13	1	1	0	0	0	0	0	0	0	0	0	.699843
12	2	1	0	0	0	0	0	0	0	0	0	.905587
11	4	0	0	0	0	0	0	0	0	0	0	.836641
11	2	2	0	0	0	0	0	0	0	0	0	1.103307
11	1	1	1	1	0	0	0	0	0	0	0	1.369974
10	4	1	0	0	0	0	0	0	0	0	0	1.158939
10	3	1	1	0	0	0	0	0	0	0	0	1.375279
10	2	1	1	1	0	0	0	0	0	0	0	1.558939
13	2	0	0	0	0	0	0	0	0	0	0	.566510
12	3	0	0	0	0	0	0	0	0	0	0	.721928
12	1	1	1	0	0	0	0	0	0	0	0	1.038921
11	3	1	0	0	0	0	0	0	0	0	0	1.052982
11	2	1	1	0	0	0	0	0	0	0	0	1.236641
10	5	0	0	0	0	0	0	0	0	0	0	.918296
10	3	2	0	0	0	0	0	0	0	0	0	1.241946
10	2	2	1	0	0	0	0	0	0	0	0	1.426605
10	1	1	1	1	1	0	0	0	0	0	0	1.692272

N=15

9	6	0	0	0	0	0	0	0	0	0	0	.970951
9	4	2	0	0	0	0	0	0	0	0	0	1.338269
9	3	3	0	0	0	0	0	0	0	0	0	1.370951
9	3	1	1	1	0	0	0	0	0	0	0	1.687943
9	2	2	1	1	0	0	0	0	0	0	0	1.738269
9	1	1	1	1	1	1	0	0	0	0	0	2.004936
8	6	1	0	0	0	0	0	0	0	0	0	1.272906
8	5	1	1	0	0	0	0	0	0	0	0	1.532915
8	4	2	1	0	0	0	0	0	0	0	0	1.640224
8	3	3	1	0	0	0	0	0	0	0	0	1.672906
8	3	2	1	1	0	0	0	0	0	0	0	1.856565
8	2	2	2	1	0	0	0	0	0	0	0	1.906891
8	2	1	1	1	1	1	0	0	0	0	0	2.173557
7	6	2	0	0	0	0	0	0	0	0	0	1.429473
7	5	3	0	0	0	0	0	0	0	0	0	1.505823
7	5	1	1	1	0	0	0	0	0	0	0	1.822816
7	4	3	1	0	0	0	0	0	0	0	0	1.746466
7	4	2	1	1	0	0	0	0	0	0	0	1.930125
7	3	3	2	0	0	0	0	0	0	0	0	1.829473
7	3	2	2	1	0	0	0	0	0	0	0	2.013132
7	3	1	1	1	1	1	0	0	0	0	0	2.279799
7	2	2	2	1	1	0	0	0	0	0	0	2.196792
6	6	3	0	0	0	0	0	0	0	0	0	1.521928
6	6	1	1	1	0	0	0	0	0	0	0	1.838921
6	5	3	1	0	0	0	0	0	0	0	0	1.781937
6	5	2	1	1	0	0	0	0	0	0	0	1.965596
6	4	4	1	0	0	0	0	0	0	0	0	1.806239
6	4	3	1	1	0	0	0	0	0	0	0	2.022580
6	4	2	1	1	1	0	0	0	0	0	0	2.206239
6	3	3	3	0	0	0	0	0	0	0	0	1.921928
6	3	3	1	1	1	0	0	0	0	0	0	2.238921
6	3	2	2	1	1	0	0	0	0	0	0	2.289246
6	2	2	2	2	1	0	0	0	0	0	0	2.339572
5	5	5	0	0	0	0	0	0	0	0	0	1.584963
5	5	3	2	0	0	0	0	0	0	0	0	1.908613
5	5	2	2	1	0	0	0	0	0	0	0	2.092272
5	5	1	1	1	1	1	0	0	0	0	0	2.358939
5	4	4	1	1	0	0	0	0	0	0	0	2.066248
5	4	3	2	1	0	0	0	0	0	0	0	2.149255
5	4	2	2	2	0	0	0	0	0	0	0	2.199581
5	4	2	1	1	1	1	0	0	0	0	0	2.466248
5	3	3	2	2	0	0	0	0	0	0	0	2.232263
5	3	3	1	1	1	1	0	0	0	0	0	2.498930
5	3	2	2	1	1	1	0	0	0	0	0	2.549255
5	2	2	2	2	1	1	0	0	0	0	0	2.599581
4	4	4	2	1	0	0	0	0	0	0	0	2.173557
4	4	3	3	1	0	0	0	0	0	0	0	2.206239
4	4	3	2	1	1	0	0	0	0	0	0	2.389898
4	4	2	2	2	1	0	0	0	0	0	0	2.440224
4	3	3	3	2	0	0	0	0	0	0	0	2.289246
4	3	3	2	2	1	0	0	0	0	0	0	2.472906
4	3	2	2	2	2	0	0	0	0	0	0	2.523231
4	2	2	2	2	2	1	0	0	0	0	0	2.706891
3	3	3	3	2	1	0	0	0	0	0	0	2.505587

9	5	1	0	0	0	0	0	0	0	0	0	1.230960
9	4	1	1	0	0	0	0	0	0	0	0	1.471602
9	3	2	1	0	0	0	0	0	0	0	0	1.554610
9	2	2	2	0	0	0	0	0	0	0	0	1.604936
9	2	1	1	1	1	0	0	0	0	0	0	1.871602
8	7	0	0	0	0	0	0	0	0	0	0	.996792
8	5	2	0	0	0	0	0	0	0	0	0	1.399581
8	4	3	0	0	0	0	0	0	0	0	0	1.456565
8	4	1	1	1	0	0	0	0	0	0	0	1.773557
8	3	2	2	0	0	0	0	0	0	0	0	1.728231
8	3	1	1	1	1	0	0	0	0	0	0	1.989898
8	2	2	1	1	1	0	0	0	0	0	0	2.040224
7	7	1	0	0	0	0	0	0	0	0	0	1.286693
7	6	1	1	0	0	0	0	0	0	0	0	1.562807
7	5	2	1	0	0	0	0	0	0	0	0	1.689482
7	4	4	0	0	0	0	0	0	0	0	0	1.530125
7	4	2	2	0	0	0	0	0	0	0	0	1.796792
7	4	1	1	1	1	0	0	0	0	0	0	2.063458
7	3	3	1	1	0	0	0	0	0	0	0	1.962807
7	3	2	1	1	1	0	0	0	0	0	0	2.146466
7	2	2	2	2	0	0	0	0	0	0	0	2.063458
7	2	2	1	1	1	1	0	0	0	0	0	2.330125
6	6	2	1	0	0	0	0	0	0	0	0	1.706587
6	5	4	0	0	0	0	0	0	0	0	0	1.565596
6	5	2	2	0	0	0	0	0	0	0	0	1.832263
6	5	1	1	1	1	0	0	0	0	0	0	2.098930
6	4	3	2	0	0	0	0	0	0	0	0	1.889246
6	4	2	2	1	0	0	0	0	0	0	0	2.072906
6	4	1	1	1	1	1	0	0	0	0	0	2.339572
6	3	3	2	1	0	0	0	0	0	0	0	2.105587
6	3	2	2	2	0	0	0	0	0	0	0	2.156913
6	3	2	1	1	1	1	0	0	0	0	0	2.422580
6	2	2	2	1	1	1	0	0	0	0	0	2.472906
5	5	4	1	0	0	0	0	0	0	0	0	1.825605
5	5	3	1	1	0	0	0	0	0	0	0	2.041946
5	5	2	1	1	1	0	0	0	0	0	0	2.225605
5	4	4	2	0	0	0	0	0	0	0	0	1.932915
5	4	3	3	0	0	0	0	0	0	0	0	1.965596
5	4	3	1	1	1	0	0	0	0	0	0	2.282589
5	4	2	2	1	1	0	0	0	0	0	0	2.332915
5	3	3	3	1	0	0	0	0	0	0	0	2.181937
5	3	3	2	1	1	0	0	0	0	0	0	2.365596
5	3	2	2	2	1	0	0	0	0	0	0	2.415922
5	2	2	2	2	2	0	0	0	0	0	0	2.466248
4	4	4	3	0	0	0	0	0	0	0	0	1.989898
4	4	4	1	1	1	0	0	0	0	0	0	2.306891
4	4	3	2	2	0	0	0	0	0	0	0	2.256565
4	4	3	1	1	1	1	0	0	0	0	0	2.523231
4	4	2	2	1	1	1	0	0	0	0	0	2.573557
4	3	3	3	1	1	0	0	0	0	0	0	2.422580
4	3	3	2	1	1	1	0	0	0	0	0	2.606239
4	3	2	2	2	1	1	0	0	0	0	0	2.656565
3	3	3	3	3	0	0	0	0	0	0	0	2.321928
3	3	3	3	1	1	1	0	0	0	0	0	2.638921

N=15

3	3	3	2	2	2	0	0	0	0	0	0	2.555913	3	3	3	2	2	1	1	0	0	0	0	0	2.889246
3	3	2	2	2	2	1	0	0	0	0	0	2.739572	3	2	2	2	2	2	2	0	0	0	0	0	2.789898

N=16

15	1	0	0	0	0	0	0	0	0	0	0	.337290	14	2	0	0	0	0	0	0	0	0	0	0	.542564
14	1	1	0	0	0	0	0	0	0	0	0	.668564	13	3	0	0	0	0	0	0	0	0	0	0	.696212
13	2	1	0	0	0	0	0	0	0	0	0	.868393	13	1	1	1	0	0	0	0	0	0	0	0	.998393
12	4	0	0	0	0	0	0	0	0	0	0	.811278	12	3	1	0	0	0	0	0	0	0	0	0	1.014098
12	2	2	0	0	0	0	0	0	0	0	0	1.061278	12	2	1	1	0	0	0	0	0	0	0	0	1.186278
12	1	1	1	1	0	0	0	0	0	0	0	1.311278	11	5	0	0	0	0	0	0	0	0	0	0	.896038
11	4	1	0	0	0	0	0	0	0	0	0	1.121641	11	3	2	0	0	0	0	0	0	0	0	0	1.199460
11	3	1	1	0	0	0	0	0	0	0	0	1.324460	11	2	2	1	0	0	0	0	0	0	0	0	1.371641
11	2	1	1	1	0	0	0	0	0	0	0	1.496641	11	1	1	1	1	1	0	0	0	0	0	0	1.621641
10	6	0	0	0	0	0	0	0	0	0	0	.954434	10	5	1	0	0	0	0	0	0	0	0	0	1.198192
10	4	2	0	0	0	0	0	0	0	0	0	1.298795	10	4	1	1	0	0	0	0	0	0	0	0	1.428795
10	3	3	0	0	0	0	0	0	0	0	0	1.329434	10	3	2	1	0	0	0	0	0	0	0	0	1.501615
10	3	1	1	1	0	0	0	0	0	0	0	1.626615	10	2	2	2	0	0	0	0	0	0	0	0	1.548795
10	2	2	1	1	0	0	0	0	0	0	0	1.673795	10	2	1	1	1	1	0	0	0	0	0	0	1.798795
10	1	1	1	1	1	1	0	0	0	0	0	1.923795	9	7	0	0	0	0	0	0	0	0	0	0	.988699
9	6	1	0	0	0	0	0	0	0	0	0	1.247556	9	5	2	0	0	0	0	0	0	0	0	0	1.366315
9	5	1	1	0	0	0	0	0	0	0	0	1.491315	9	4	3	0	0	0	0	0	0	0	0	0	1.419737
9	4	2	1	0	0	0	0	0	0	0	0	1.591917	9	4	1	1	1	0	0	0	0	0	0	0	1.716917
9	3	3	1	0	0	0	0	0	0	0	0	1.622556	9	3	2	2	0	0	0	0	0	0	0	0	1.669737
9	3	2	1	1	0	0	0	0	0	0	0	1.794737	9	3	1	1	1	1	0	0	0	0	0	0	1.919737
9	2	2	2	1	0	0	0	0	0	0	0	1.841917	9	2	2	1	1	1	0	0	0	0	0	0	1.966917
9	2	1	1	1	1	1	0	0	0	0	0	2.091917	9	1	1	1	1	1	1	1	0	0	0	0	2.216917
8	8	0	0	0	0	0	0	0	0	0	0	1.000000	8	7	1	0	0	0	0	0	0	0	0	0	1.271782
8	6	2	0	0	0	0	0	0	0	0	0	1.405639	8	6	1	1	0	0	0	0	0	0	0	0	1.530639
8	5	3	0	0	0	0	0	0	0	0	0	1.477217	8	5	2	1	0	0	0	0	0	0	0	0	1.649397
8	5	1	1	1	0	0	0	0	0	0	0	1.774397	8	4	4	0	0	0	0	0	0	0	0	0	1.500000
8	4	3	1	0	0	0	0	0	0	0	0	1.702820	8	4	2	2	0	0	0	0	0	0	0	0	1.750000
8	4	2	1	1	0	0	0	0	0	0	0	1.875000	8	4	1	1	1	1	0	0	0	0	0	0	2.000000
8	3	3	2	0	0	0	0	0	0	0	0	1.780639	8	3	3	1	1	0	0	0	0	0	0	0	1.905639
8	3	2	2	1	0	0	0	0	0	0	0	1.952820	8	3	2	1	1	1	0	0	0	0	0	0	2.077820
8	3	1	1	1	1	1	0	0	0	0	0	2.202820	8	2	2	2	2	0	0	0	0	0	0	0	2.000000
8	2	2	2	1	1	0	0	0	0	0	0	2.125000	8	2	2	1	1	1	1	0	0	0	0	0	2.250000
8	2	1	1	1	1	1	1	0	0	0	0	2.375000	7	7	2	0	0	0	0	0	0	0	0	0	1.418564
7	7	1	1	0	0	0	0	0	0	0	0	1.543564	7	6	3	0	0	0	0	0	0	0	0	0	1.506241
7	6	2	1	0	0	0	0	0	0	0	0	1.677421	7	6	1	1	1	0	0	0	0	0	0	0	1.802421
7	5	4	0	0	0	0	0	0	0	0	0	1.546180	7	5	3	1	0	0	0	0	0	0	0	0	1.748999
7	5	2	2	0	0	0	0	0	0	0	0	1.796180	7	5	2	1	1	0	0	0	0	0	0	0	1.921180
7	5	1	1	1	1	0	0	0	0	0	0	2.046180	7	4	4	1	0	0	0	0	0	0	0	0	1.771782
7	4	3	2	0	0	0	0	0	0	0	0	1.849602	7	4	3	1	1	0	0	0	0	0	0	0	1.974602
7	4	2	2	1	0	0	0	0	0	0	0	2.021782	7	4	2	1	1	1	0	0	0	0	0	0	2.146782
7	4	1	1	1	1	1	0	0	0	0	0	2.271782	7	3	3	3	0	0	0	0	0	0	0	0	1.880241
7	3	3	2	1	0	0	0	0	0	0	0	2.052421	7	3	3	1	1	1	0	0	0	0	0	0	2.177421
7	3	2	2	2	0	0	0	0	0	0	0	2.099602	7	3	2	2	1	1	0	0	0	0	0	0	2.224602
7	3	2	1	1	1	1	0	0	0	0	0	2.349602	7	3	1	1	1	1	1	1	0	0	0	0	2.474602
7	2	2	2	2	1	0	0	0	0	0	0	2.271782	7	2	2	2	1	1	1	0	0	0	0	0	2.396782
7	2	2	1	1	1	1	1	0	0	0	0	2.521782	6	6	4	0	0	0	0	0	0	0	0	0	1.561278
6	6	3	1	0	0	0	0	0	0	0	0	1.764098	6	6	2	2	0	0	0	0	0	0	0	0	1.811278
6	6	2	1	1	0	0	0	0	0	0	0	1.936278	6	6	1	1	1	1	0	0	0	0	0	0	2.061278
6	5	5	0	0	0	0	0	0	0	0	0	1.579434	6	5	4	1	0	0	0	0	0	0	0	0	1.806037

N=16

6	5	3	2	0	0	0	0	0	0	0	0	1.882856	6	5	3	1	1	0	0	0	0	0	0	0	2.007856
6	5	2	2	1	0	0	0	0	0	0	0	2.055037	6	5	2	1	1	1	0	0	0	0	0	0	2.180037
6	5	1	1	1	1	1	0	0	0	0	0	2.305037	6	4	4	2	0	0	0	0	0	0	0	0	1.905639
6	4	4	1	1	0	0	0	0	0	0	0	2.030639	6	4	3	3	0	0	0	0	0	0	0	0	1.936278
6	4	3	2	1	0	0	0	0	0	0	0	2.108459	6	4	3	1	1	1	0	0	0	0	0	0	2.233459
6	4	2	2	2	0	0	0	0	0	0	0	2.155639	6	4	2	2	1	1	0	0	0	0	0	0	2.280639
6	4	2	1	1	1	1	0	0	0	0	0	2.405639	6	4	1	1	1	1	1	1	0	0	0	0	2.530639
6	3	3	3	1	0	0	0	0	0	0	0	2.139098	6	3	3	2	2	0	0	0	0	0	0	0	2.186278
6	3	3	2	1	1	0	0	0	0	0	0	2.311278	6	3	3	1	1	1	1	0	0	0	0	0	2.436278
6	3	2	2	2	1	0	0	0	0	0	0	2.358459	6	3	2	2	1	1	1	0	0	0	0	0	2.483459
6	3	2	1	1	1	1	1	0	0	0	0	2.608459	6	2	2	2	2	2	0	0	0	0	0	0	2.405639
6	2	2	2	2	1	1	0	0	0	0	0	2.530639	6	2	2	2	1	1	1	1	0	0	0	0	2.655639
5	5	5	1	0	0	0	0	0	0	0	0	1.823192	5	5	4	2	0	0	0	0	0	0	0	0	1.928795
5	5	4	1	1	0	0	0	0	0	0	0	2.048795	5	5	3	3	0	0	0	0	0	0	0	0	1.954434
5	5	3	2	1	0	0	0	0	0	0	0	2.126615	5	5	3	1	1	1	0	0	0	0	0	0	2.251615
5	5	2	2	2	0	0	0	0	0	0	0	2.173795	5	5	2	2	1	1	0	0	0	0	0	0	2.298795
5	5	2	1	1	1	1	0	0	0	0	0	2.423795	5	5	1	1	1	1	1	1	0	0	0	0	2.548795
5	4	4	3	0	0	0	0	0	0	0	0	1.977217	5	4	4	2	1	0	0	0	0	0	0	0	2.149397
5	4	4	1	1	1	0	0	0	0	0	0	2.274397	5	4	3	3	1	0	0	0	0	0	0	0	2.180037
5	4	3	2	2	0	0	0	0	0	0	0	2.227217	5	4	3	2	1	1	0	0	0	0	0	0	2.352217
5	4	3	1	1	1	1	0	0	0	0	0	2.477217	5	4	2	2	2	1	0	0	0	0	0	0	2.399397
5	4	2	2	1	1	1	0	0	0	0	0	2.524397	5	4	2	1	1	1	1	1	0	0	0	0	2.649397
5	3	3	3	2	0	0	0	0	0	0	0	2.257856	5	3	3	3	1	1	0	0	0	0	0	0	2.382856
5	3	3	2	2	1	0	0	0	0	0	0	2.430037	5	3	3	2	1	1	1	0	0	0	0	0	2.555037
5	3	3	1	1	1	1	1	0	0	0	0	2.680037	5	3	2	2	2	2	0	0	0	0	0	0	2.477217
5	3	2	2	2	1	1	0	0	0	0	0	2.602217	5	3	2	2	1	1	1	1	0	0	0	0	2.727217
5	2	2	2	2	2	1	0	0	0	0	0	2.649397	5	2	2	2	2	1	1	1	0	0	0	0	2.774397
4	4	4	4	0	0	0	0	0	0	0	0	2.000000	4	4	4	3	1	0	0	0	0	0	0	0	2.202820
4	4	4	2	2	0	0	0	0	0	0	0	2.250000	4	4	4	2	1	1	0	0	0	0	0	0	2.375000
4	4	4	1	1	1	1	0	0	0	0	0	2.500000	4	4	3	3	2	0	0	0	0	0	0	0	2.280639
4	4	3	3	1	1	0	0	0	0	0	0	2.405639	4	4	3	2	2	1	0	0	0	0	0	0	2.452820
4	4	3	2	1	1	1	0	0	0	0	0	2.577820	4	4	3	1	1	1	1	1	0	0	0	0	2.702820
4	4	2	2	2	2	0	0	0	0	0	0	2.500000	4	4	2	2	2	1	1	0	0	0	0	0	2.625000
4	4	2	2	1	1	1	1	0	0	0	0	2.750000	4	3	3	3	3	0	0	0	0	0	0	0	2.311278
4	3	3	3	2	1	0	0	0	0	0	0	2.483459	4	3	3	3	1	1	1	0	0	0	0	0	2.608459
4	3	3	2	2	2	0	0	0	0	0	0	2.530639	4	3	3	2	2	1	1	0	0	0	0	0	2.655639
4	3	3	2	1	1	1	1	0	0	0	0	2.780639	4	3	2	2	2	2	1	0	0	0	0	0	2.702820
4	3	2	2	2	1	1	1	0	0	0	0	2.827820	4	2	2	2	2	2	2	0	0	0	0	0	2.750000
4	2	2	2	2	2	1	1	0	0	0	0	2.875000	3	3	3	3	3	1	0	0	0	0	0	0	2.514098
3	3	3	3	2	2	0	0	0	0	0	0	2.561278	3	3	3	3	2	1	1	0	0	0	0	0	2.686278
3	3	3	3	1	1	1	1	0	0	0	0	2.811278	3	3	3	2	2	2	1	0	0	0	0	0	2.733459
3	3	3	2	2	1	1	1	0	0	0	0	2.858459	3	3	2	2	2	2	2	0	0	0	0	0	2.780639
3	3	2	2	2	2	1	1	0	0	0	0	2.905639	3	2	2	2	2	2	2	1	0	0	0	0	2.952820
2	2	2	2	2	2	2	2	0	0	0	0	3.000000	0	0	0	0	0	0	0	0	0	0	0	0	.000000

NUMBER OF PARTITIONS

The number of ways in which a particular number n (earlier denoted by capital N) can be partitioned grows pretty fast with increasing n. To give an idea of this growth the entries up to n = 52 of a famous table of partitions follows*.

The number of ways a number n can be partitioned into exactly m parts is usually denoted by p(n,m). For instance (n = 4, m = 2):

$$p(4,2) = 2,$$

for there are the two partitions (3,1) and (2,2) which represent 4 as the sum of exactly two parts. In order to find this number in the following table enter at the column headed by n, and read the desired number at the intersection of this column with row m. E.g. p(24,7) = 201. (For values of m>1/2n see equation (3) of the introductory note by Todd to his Table.)

The total number of ways a number n can be partitioned is, of course, the sum of the number of partitions of n into exactly m parts taken for all m = 1→n. This is called the number of unrestricted partitions of n, usually denoted by

$$p(n) = \sum_{m=1}^{m=n} p(n,m)$$

For instance, for n = 4 we have the following partitions: (4); (3,1); (2,2); (3,1,1); (1,1,1,1); hence

$$p(4) = p(4,1)+p(4,2)+P(4,3)+p(4,4) = 1 + 2 + 1 + 1 = 5$$

These numbers are given in the block at the end of this Table.

The observant reader may have noticed that for each number n the number of ways this number can be partitioned first increases with increasing number of parts, reaches a maximum for a particular m, call this $\bar{m}$, and then decreases when more parts are considered ($m>\bar{m}$). For instance, for n = 24, $\bar{m}$ = 7, and p(24,7) = 201. This means that an observer watching 24 people who may freely assemble to form groups will see them most of the time (exactly 12.76% of the time, for p(24) = 1575) being assembled in seven groups.

*Todd, J.A.: "A Table of Partitions", Proc. London Soc. Math. (2) 48, 229-242 (1943).

A TABLE OF PARTITIONS

By J. A. Todd.

[Received 10 December, 1940.—Read 21 May, 1942.]

1. Let $p(n)$ be the number of unrestricted partitions of the positive integer n, and let $p(n, m)$ denote the number of partitions of n into exactly m parts. Then

$$p(n, 1) = 1 \qquad (1)$$

and, for $m > 1$,

$$p(n, m) = p(n-1, m-1)+p(n-m, m), \qquad (2)$$

since the first term on the right of (2) represents the number of partitions of n into m parts, one at least of which is unity, and the second term represents the number of partitions of n into m parts, each of which exceeds unity. By using this formula recursively the values of $p(n, m)$ may be calculated for each pair of integers n, m. This leads to Euler's table of double entry, described by Chrystal†.

It is clear that $p(n, m) = 0$ if $m > n$, and that, if $m \geqslant \frac{1}{2}n$,

$$p(n, m) = p(n-m). \qquad (3)$$

Thus it is sufficient to tabulate the values of $p(n, m)$ for $m \leqslant \frac{1}{2}n$.

The present table contains the values of $p(n, m)$ for $1 \leqslant n \leqslant 100$, $1 \leqslant m \leqslant [\frac{1}{2}(n+1)]$, the final entry, for each value of n, being $p[\frac{1}{2}(n+1)]$ corresponding to the value $m = [\frac{1}{2}(n+1)]$. This final entry is marked

† G. Chrystal, *Algebra*, 2 (1900), 557–561. Euler's table [*Opera* (1), 8 (1922), 327–328] gives $p(n+m, m)$ for $1 \leqslant m \leqslant 11$, $1 \leqslant n \leqslant 69$.

with an asterisk to indicate that it and the unentered values of $p(n, m)$ for larger values of m are obtainable from a table of unrestricted partitions by means of formula (3). For this purpose an extract from Gupta's table up to $n = 600$ is given on p. 233. The values of $p(n, m)$ were obtained from the formula

$$p(n, m) = \sum_{r=1}^{m} p(n-m, r), \tag{4}$$

which is an immediate consequence of (2). Since

$$\sum_{m=1}^{n} p(n, m) = p(n), \tag{5}$$

a check on the work is provided by the formula

$$\sum_{m=1}^{s} p(n, m) = p(n) - \sum_{r=0}^{n-s-1} p(r), \tag{6}$$

where $s = [\frac{1}{2}(n+1)]$, the expression on the right being computed from the values of $p(n)$ tabulated by Gupta†.

2. It is easily seen that the function

$$f_m(x) = \frac{x^m}{(1-x)(1-x^2)\dots(1-x^m)} \tag{7}$$

is the generating function of $p(n, m)$. It is easily seen from (2) that for fixed m, $p(n, m)$ behaves on the average like a polynomial of degree $m-1$ in n, in which the dominant term is

$$\frac{n^{m-1}}{(m-1)!\, m!}, \tag{8}$$

an expression which gives the approximate order of magnitude of $p(n, m)$ when n is large compared with m. Thus (8) gives the order of magnitude of $p(n, m)$ for fixed m, when n is large.

Trinity College,
Cambridge.

† H. Gupta, *Proc. London Math. Soc.* (2), 39 (1935), 142–149. We define, conventionally, $p(0) = 1$.

m \ n	1	2	3	4	5	6	7	8	9	10
1	*1	*1	1	1	1	1	1	1	1	1
2			*1	*2	2	3	3	4	4	5
3					*2	*3	4	5	7	8
4							*3	*5	6	9
5									*5	*7

m \ n	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1
2	5	6	6	7	7	8	8	9	9	10
3	10	12	14	16	19	21	24	27	30	33
4	11	15	18	23	27	34	39	47	54	64
5	10	13	18	23	30	37	47	57	70	84
6	*7	*11	14	20	26	35	44	58	71	90
7			*11	*15	21	28	38	49	65	82
8					*15	*22	29	40	52	70
9							*22	*30	41	54
10									*30	*42

m \ n	21	22	23	24	25	26	27	28	29	30
1	1	1	1	1	1	1	1	1	1	1
2	10	11	11	12	12	13	13	14	14	15
3	37	40	44	48	52	56	61	65	70	75
4	72	84	94	108	120	136	150	169	185	206
5	101	119	141	164	192	221	255	291	333	377
6	110	136	163	199	235	282	331	391	454	532
7	105	131	164	201	248	300	364	436	522	618
8	89	116	146	186	230	288	352	434	525	638
9	73	94	123	157	201	252	318	393	488	598
10	55	75	97	128	164	212	267	340	423	530
11	*42	*56	76	99	131	169	219	278	355	445
12			*56	*77	100	133	172	224	285	366
13					*77	*101	134	174	227	290
14							*101	*135	175	229
15									*135	*176

m \ n	31	32	33	34	35	36	37	38
1	1	1	1	1	1	1	1	1
2	15	16	16	17	17	18	18	19
3	80	85	91	96	102	108	114	120
4	225	249	270	297	321	351	378	411
5	427	480	540	603	674	748	831	918
6	612	709	811	931	1057	1206	1360	1540
7	733	860	1009	1175	1367	1579	1824	2093
8	764	919	1090	1297	1527	1801	2104	2462
9	732	887	1076	1291	1549	1845	2194	2592
10	653	807	984	1204	1455	1761	2112	2534
11	560	695	863	1060	1303	1586	1930	2331
12	460	582	725	905	1116	1380	1686	2063
13	373	471	597	747	935	1158	1436	1763
14	293	378	478	608	762	957	1188	1478
15	230	295	381	483	615	773	972	1210
16	*176	*231	296	383	486	620	780	983
17			*231	*297	384	488	623	785
18					*297	*385	489	625
19							*385	*490

m \ n	39	40	41	42	43	44	45	46
1	1	1	1	1	1	1	1	1
2	19	20	20	21	21	22	22	23
3	127	133	140	147	154	161	169	176
4	441	478	511	551	588	632	672	720
5	1014	1115	1226	1342	1469	1602	1747	1898
6	1729	1945	2172	2432	2702	3009	3331	3692
7	2400	2738	3120	3539	4011	4526	5102	5731
8	2857	3319	3828	4417	5066	5812	6630	7564
9	3060	3589	4206	4904	5708	6615	7657	8824
10	3015	3590	4242	5013	5888	6912	8070	9418
11	2812	3370	4035	4802	5708	6751	7972	9373
12	2503	3036	3655	4401	5262	6290	7476	8877
13	2164	2637	3210	3882	4691	5635	6761	8073
14	1819	2241	2738	3345	4057	4920	5928	7139
15	1508	1861	2297	2815	3446	4192	5096	6158
16	1225	1530	1891	2339	2871	3523	4293	5231
17	990	1236	1545	1913	2369	2913	3579	4370
18	788	995	1243	1556	1928	2391	2943	3621
19	626	790	998	1248	1563	1939	2406	2965
20	*490	*627	791	1000	1251	1568	1946	2417
21			*627	*792	1001	1253	1571	1951
22					*792	*1002	1254	1573
23							*1002	*1255

m \ n	47	48	49	50	51	52
1	1	1	1	1	1	1
2	23	24	24	25	25	26
3	184	192	200	208	217	225
4	764	816	864	920	972	1033
5	2062	2233	2418	2611	2818	3034
6	4070	4494	4935	5427	5942	6510
7	6430	7190	8033	8946	9953	11044
8	8588	9749	11018	12450	14012	15765
9	10156	11648	13338	15224	17354	19720
10	10936	12690	14663	16928	19466	22367
11	11004	12866	15021	17475	20298	23501
12	10489	12384	14552	17084	19978	23334
13	9624	11424	13542	15988	18847	22142
14	8551	10232	12186	14499	17176	20325
15	7434	8932	10715	12801	15272	18148
16	6334	7665	9228	11098	13287	15892
17	5332	6469	7841	9459	11395	13671
18	4426	5409	6570	7976	9635	11626
19	3651	4468	5465	6647	8077	9770
20	2980	3673	4498	5507	6703	8154
21	2424	2991	3688	4520	5537	6745
22	1954	2429	2998	3699	4535	5559
23	1574	1956	2432	3003	3706	4546
24	*1255	*1575	1957	2434	3006	3711
25			*1575	*1958	2435	3008
26					*1958	*2436

n	p(n)	n	p(n)	n	p(n)	n	p(n)	n	p(n)
1	1	11	56	21	792	31	6842	41	44583
2	2	12	77	22	1002	32	8349	42	53174
3	3	13	101	23	1255	33	10143	43	63261
4	5	14	135	24	1575	34	12310	44	75175
5	7	15	176	25	1958	35	14883	45	89134
6	11	16	231	26	2436	36	17977	46	1 05558
7	15	17	297	27	3010	37	21637	47	1 24754
8	22	18	385	28	3718	38	26015	48	1 47273
9	30	19	490	29	4565	39	31185	49	1 73525
10	42	20	627	30	5604	40	37338	50	2 04226

Coalitions

Consider the case in which the elements distributed over the parts in a particular partition are distinguishable or, what amounts to the same, can be given individual names. For example, consider the particular partition (2,2), i.e., one of the two possible partitions of 4 into exactly two parts. If the elements are indistinguishable (or their distinction is ignored) we have

● ●   ● ●

Should we, however, become cognizant of individual distinctions of these elements (we can "name" them) then we have exactly

	AB	CD,
or	AC	BD,
or	AD	BC,

that is, exactly three distinct representations of the partition (2,2). For obvious reasons, each of these distinct representations is called a "coalition", suggesting that the elements formed in each group have come together to act in concert either independently of, or competitively with, the other groups. A typical case in question is the number of different ways four Bridge players, A,B,C,D, may arrange their partnerships (coalitions) so as to realize the full variety of their playing potential (as we have seen, this can be done in exactly three different ways).

Consider now the other one of the two possible partitions of 4 into exactly 2 parts, namely (3,1):

● ● ●   ●

Again, should we be able (or willing) to distinguish between the elements distributed over this partition, we have

ABC	D,
ABD	C,
ACD	B,
BCD	A,

that is, four distinct coalitions. Considering this and the previous case together, we may say that the number of ways in which four distinct elements can distribute themselves into exactly two groups is seven. In general, we shall denote the number of coalitions of n distinct elements forming m groups by

$$S(n,m)$$

and the number of unrestricted coalitions by

$$S(n) = \sum_{m=1}^{m=n} S(n,m).$$

The capital letter "S" for denoting these numbers was chosen to honor the Scottish mathematician James Stirling (1692-1770), a friend of Isaak Newton, who discovered these numbers in his studies of truncated factorials. The numbers S(n,m) and S(n) are called respectively "Stirling Numbers of the Second Kind" and "Unrestricted Stirling Numbers of the Second Kind"*)

The need for having an extensive table of S(n,m) arose through Gotthard Günther's work on combinatorial possibilities of non-Aristotelian logics that admit more than the two classical values of "True" and "False". Alex Andrew computed the Table of Stirling Numbers of the Second Kind" of which only the entries up to n = 32 are reproduced on the following pages.†)

Andrew used the letter k, rather than m, to denote the number of groups, hence the expression S(n,k) to be found in the table (n and k are given in capitals). In order to find the number of coalitions for N elements assembled in K groups one enters the table headed by N, finds the row indicated by K and reads S(N,K) to the right.

---

*See Riordan, J.: An Introduction to Combinatorial Analysis, John Wiley & Sons, New York, 244 pp. (1958). Should the reader be interested in knowing the number of coalitions that are possible when a particular partition is given (say, as above, (2,2) → 3, (3,1) → 4) he would need to consult Bell Polynomials. A table for these numbers can be found in the reference above (on page 49) for n up to eight and for n up to ten in Riordan, J.: Combinatorial Identities. John Wiley & Sons, New York, 256 pp. (1968).

†Andrew, A.: "Stirling Numbers of the Second Kind". T.R.6, AF-OSR 7-64, Biological Computer Laboratory, Department of Electrical Engineering, Univsersity of Illinois, Urbana, Illinois, 151 pp. (1965). In this Table S(N,K) and S(N) are computed for all values of N≤100 that lead to numbers S(N,K) or S(N) which do not exceed $10^{109}-1$.

○ ○   ○

Due to the fact that the machine could not handle the computation of the extraordinarily large numbers that arose, they had to be computed in batches. This resulted in empty spaces in the print-out which are to read as zeros "0". For instance we find for

S(20,9) = 12 11282644725
↑

which should be read

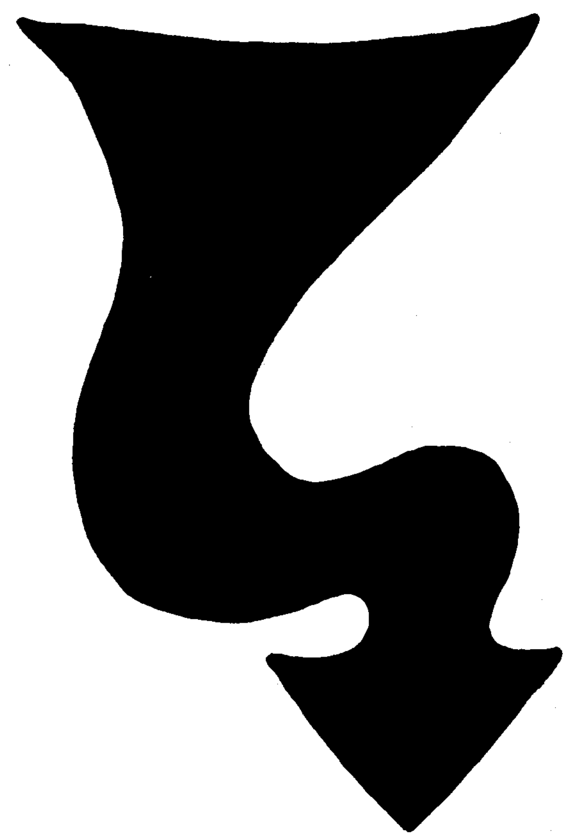
S(20,9) = 12011282644725.
↑

The observant reader may have noticed that for each number N the number of coalition first increases with increasing number of groups, reaches a maximum for a particular K, call this $\overline{K}$, and then decreases when more groups are considered ($K>\overline{K}$). For instance, for $N = 24$, $\overline{K} = 9$ and S(24,9) = 120,622,574,326,072,500. This means that an observer watching 24 people who may freely assemble to form groups will see them most of the time (exactly 27.05% of the time, for S(24) = 445,958,869,294,805,289) being assembled in nine groups.

At an earlier occasion, when numbers of partitions were discussed, a different observer watching the same 24 people came to a quite different conclusion as to their group behavior. According to him they were spending most of their time (exactly 12.76%) being assembled in seven groups.

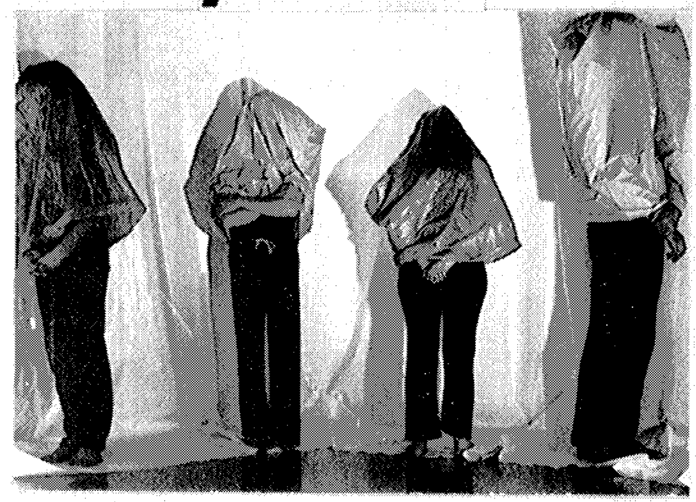
The contemplative reader may ponder over the blatant discrepancy in the descriptions by two observers of the apparently most primitive properties of the same universe, properties that are established merely by counting. What could be more objective?

TABLE OF THE STIRLING NUMBERS OF THE SECOND KIND



$p(4,1)=1$

$p(4,4)=1$

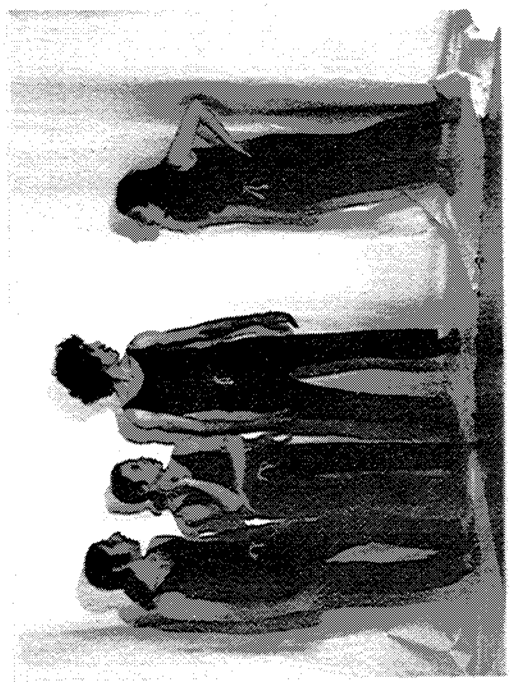


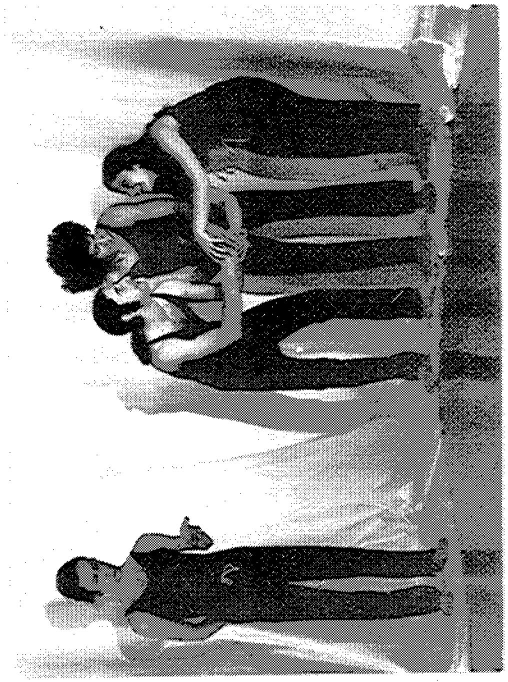
$S(4,1)=1$

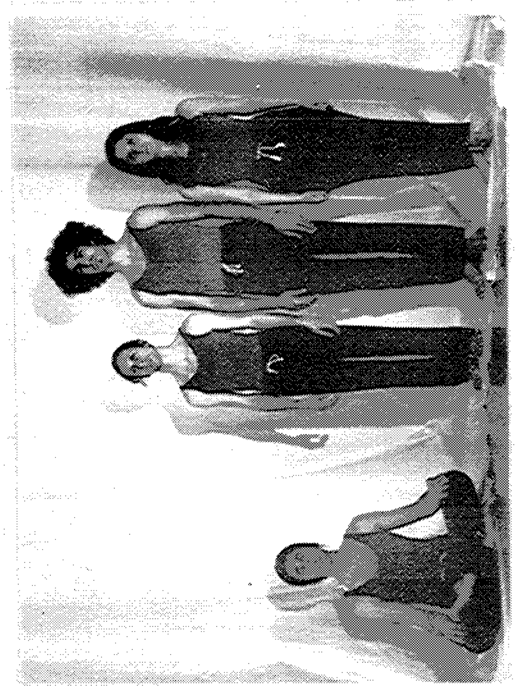
$S(4,4)=1$

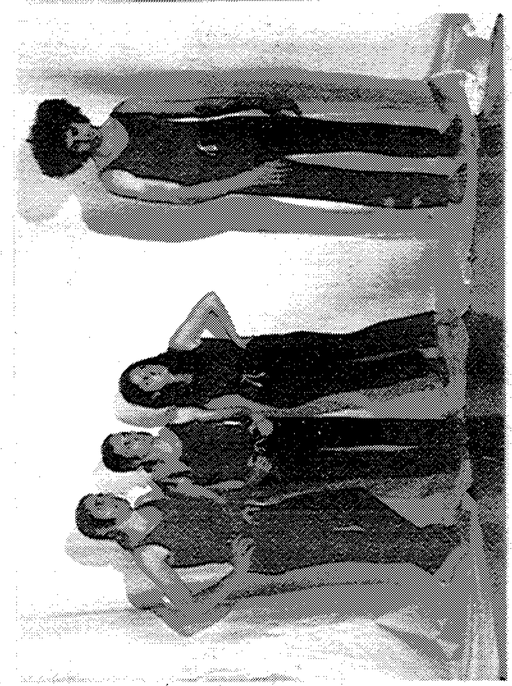
$P(4,2) = 2$

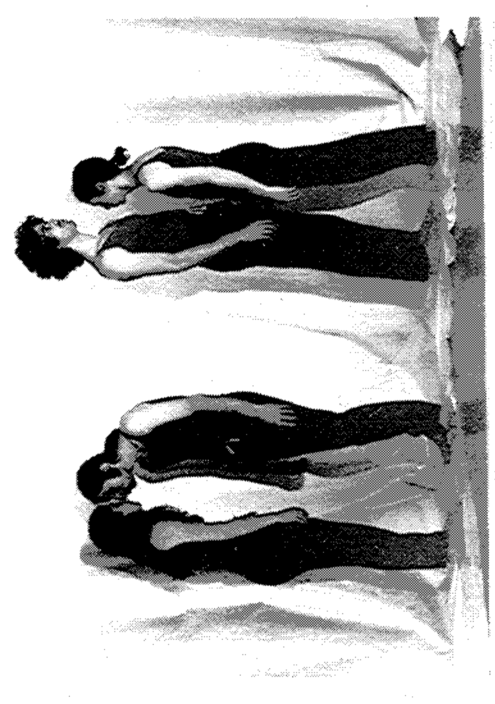
$S(4,2) = 7$





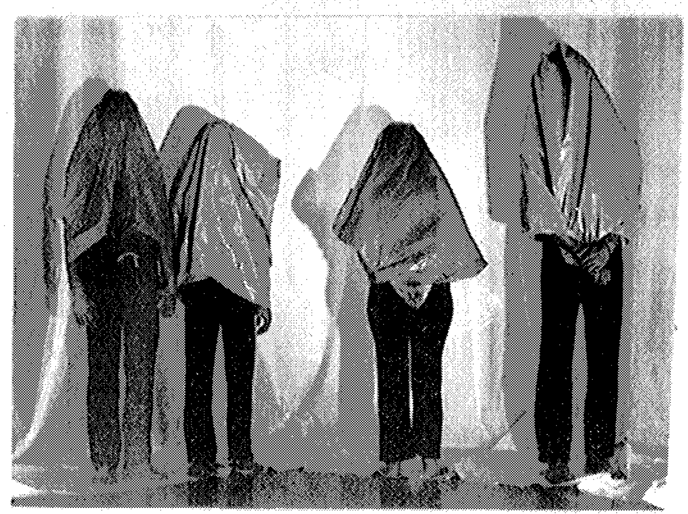


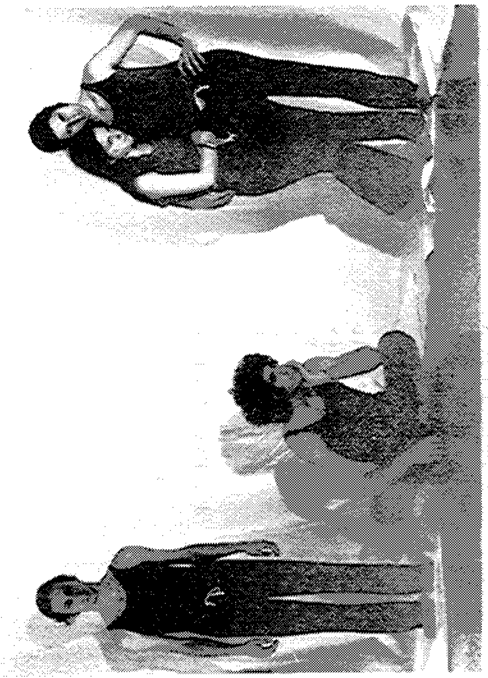


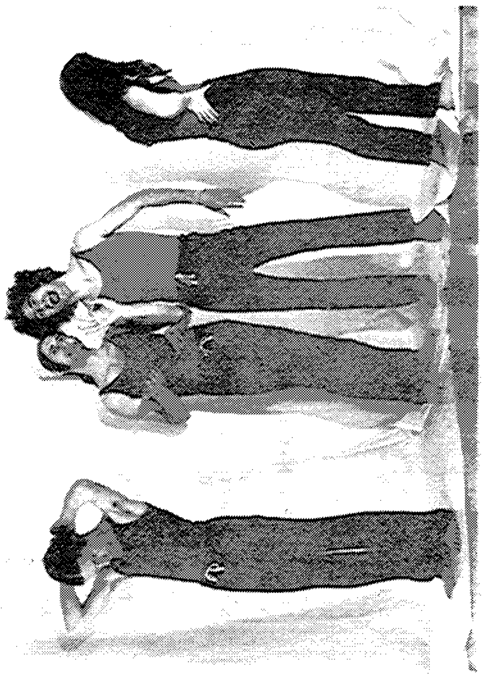


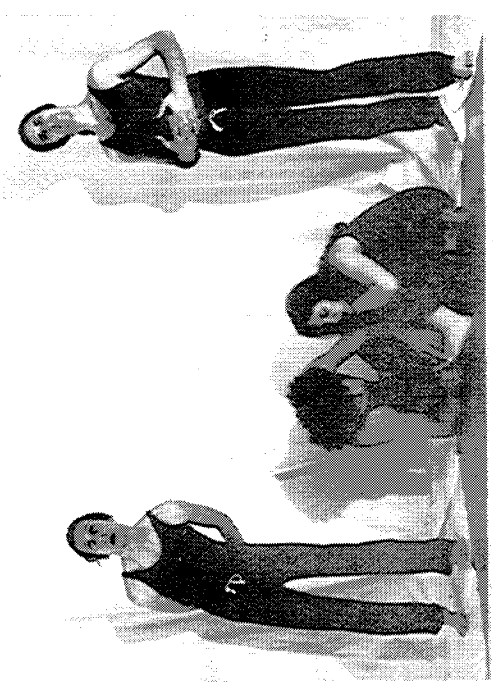
$P(4,3) = 1$

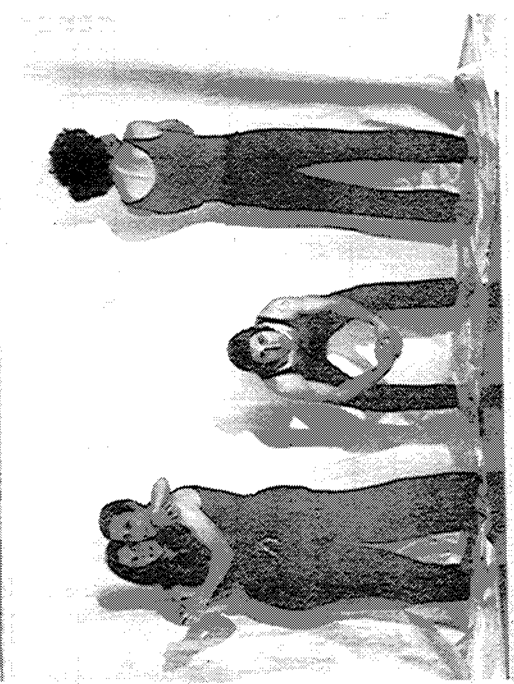
$S(4,3) = 6$

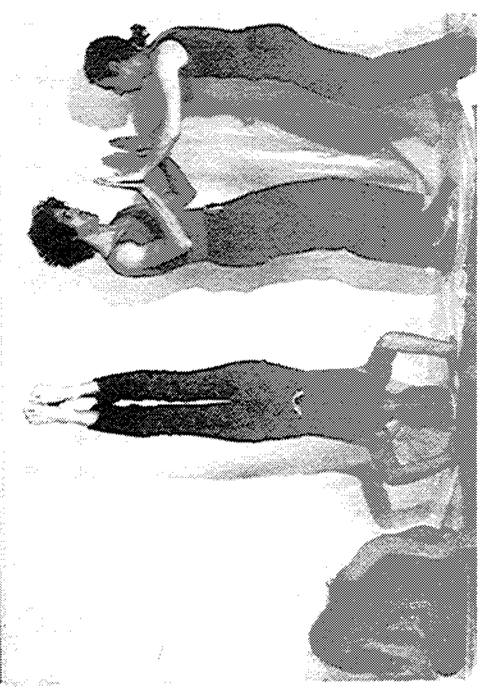












S(N,K)

```
S(N,K) FOR N =   1
K=  1              0         0         0         0         1
FOR N=  1 , MAX IS FOR K=  1
SIGMA =
          0        0         0         0         0         1
```

```
S(N,K) FOR N =   3
K=  1              0         0         0         0         1
K=  2              0         0         0         0         3
K=  3              0         0         0         0         1
FOR N=  3 , MAX IS FOR K=  2
SIGMA =
          0        0         0         0         0         5
```

```
S(N,K) FOR N =   5
K=  1              0         0         0         0         1
K=  2              0         0         0         0        15
K=  3              0         0         0         0        25
K=  4              0         0         0         0        10
K=  5              0         0         0         0         1
FOR N=  5 , MAX IS FOR K=  3
SIGMA =
          0        0         0         0         0        52
```

```
S(N,K) FOR N =   7
K=  1              0         0         0         0         1
K=  2              0         0         0         0        63
K=  3              0         0         0         0       301
K=  4              0         0         0         0       350
K=  5              0         0         0         0       140
K=  6              0         0         0         0        21
K=  7              0         0         0         0         1
FOR N=  7 , MAX IS FOR K=  4
SIGMA =
          0        0         0         0         0       877
```

```
S(N,K) FOR N =   2
K=  1              0         0         0         0         1
K=  2              0         0         0         0         1
FOR N=  2 , MAX IS FOR K=  2
SIGMA =
          0        0         0         0         0         2
```

```
S(N,K) FOR N =   4
K=  1              0         0         0         0         1
K=  2              0         0         0         0         7
K=  3              0         0         0         0         6
K=  4              0         0         0         0         1
FOR N=  4 , MAX IS FOR K=  2
SIGMA =
          0        0         0         0         0        15
SUM OF FIRST   2 TERMS=
          0        0         0         0         0        14
```

```
S(N,K) FOR N =   6
K=  1              0         0         0         0         1
K=  2              0         0         0         0        31
K=  3              0         0         0         0        90
K=  4              0         0         0         0        65
K=  5              0         0         0         0        15
K=  6              0         0         0         0         1
FOR N=  6 , MAX IS FOR K=  3
SIGMA =
          0        0         0         0         0       203
```

```
S(N,K) FOR N =   8
K=  1              0         0         0         0         1
K=  2              0         0         0         0       127
K=  3              0         0         0         0       966
K=  4              0         0         0         0      1701
K=  5              0         0         0         0      1050
K=  6              0         0         0         0       266
K=  7              0         0         0         0        28
K=  8              0         0         0         0         1
FOR N=  8 , MAX IS FOR K=  4
SIGMA =
          0        0         0         0         0      4140
```

```
S(N,K) FOR N =    9
K=  1            0        0        0        0           1
K=  2            0        0        0        0         255
K=  3            0        0        0        0        3025
K=  4            0        0        0        0        7770
K=  5            0        0        0        0        6951
K=  6            0        0        0        0        2646
K=  7            0        0        0        0         462
K=  8            0        0        0        0          36
K=  9            0        0        0        0           1
FOR N=  9 , MAX IS FOR K=  4
SIGMA -
        0        0        0        0        0       21147
SUM OF FIRST   3 TERMS=
        0        0        0        0        0       11051

S(N,K) FOR N =  10
K=  1            0        0        0        0           1
K=  2            0        0        0        0         511
K=  3            0        0        0        0        9330
K=  4            0        0        0        0       34105
K=  5            0        0        0        0       42525
K=  6            0        0        0        0       22827
K=  7            0        0        0        0        5880
K=  8            0        0        0        0         750
K=  9            0        0        0        0          45
K= 10            0        0        0        0           1
FOR N= 10 , MAX IS FOR K=  5
SIGMA =
        0        0        0        0        0      115975

S(N,K) FOR N =  11
K=  1            0        0        0        0           1
K=  2            0        0        0        0        1023
K=  3            0        0        0        0       28501
K=  4            0        0        0        0      145750
K=  5            0        0        0        0      246730
K=  6            0        0        0        0      179487
K=  7            0        0        0        0       63987
K=  8            0        0        0        0       11880
K=  9            0        0        0        0        1155
K= 10            0        0        0        0          55
K= 11            0        0        0        0           1
FOR N= 11 , MAX IS FOR K=  5
SIGMA =
        0        0        0        0        0      678570

S(N,K) FOR N =  12
K=  1            0        0        0        0           1
K=  2            0        0        0        0        2047
K=  3            0        0        0        0       86526
K=  4            0        0        0        0      611501
K=  5            0        0        0        0     1379400
K=  6            0        0        0        0     1323652
K=  7            0        0        0        0      627396
K=  8            0        0        0        0      159027
K=  9            0        0        0        0       22275
K= 10            0        0        0        0        1705
K= 11            0        0        0        0          66
K= 12            0        0        0        0           1
FOR N= 12 , MAX IS FOR K=  5
SIGMA =
        0        0        0        0        0     4213597

S(N,K) FOR N =  13
K=  1            0        0        0        0           1
K=  2            0        0        0        0        4095
K=  3            0        0        0        0      261625
K=  4            0        0        0        0     2532530
K=  5            0        0        0        0     7508501
K=  6            0        0        0        0     9321312
K=  7            0        0        0        0     5715424
K=  8            0        0        0        0     1899612
K=  9            0        0        0        0      359502
K= 10            0        0        0        0       39325
K= 11            0        0        0        0        2431
K= 12            0        0        0        0          78
K= 13            0        0        0        0           1
FOR N= 13 , MAX IS FOR K=  6
SIGMA =
        0        0        0        0        0    27644437

S(N,K) FOR N =  14
K=  1            0        0        0        0           1
K=  2            0        0        0        0        8191
K=  3            0        0        0        0      788970
K=  4            0        0        0        0    10391745
K=  5            0        0        0        0    40075035
K=  6            0        0        0        0    63436373
K=  7            0        0        0        0    49329280
K=  8            0        0        0        0    20912320
K=  9            0        0        0        0     5135130
K= 10            0        0        0        0      752752
K= 11            0        0        0        0       66066
K= 12            0        0        0        0        3367
K= 13            0        0        0        0          91
K= 14            0        0        0        0           1
FOR N= 14 , MAX IS FOR K=  6
SIGMA =
        0        0        0        0        0   190899322
```

S(N,K) FOR N = 15

K= 1		0	0	0	0	1
K= 2		0	0	0	0	16383
K= 3		0	0	0	0	2375101
K= 4		0	0	0	0	42355950
K= 5		0	0	0	0	210766920
K= 6		0	0	0	0	420693273
K= 7		0	0	0	0	408741333
K= 8		0	0	0	0	216627840
K= 9		0	0	0	0	67128490
K= 10		0	0	0	0	12662650
K= 11		0	0	0	0	1479478
K= 12		0	0	0	0	106470
K= 13		0	0	0	0	4550
K= 14		0	0	0	0	105
K= 15		0	0	0	0	1

FOR N= 15 , MAX IS FOR K= 6

SIGMA =	0	0	0	0	0	1382958545

S(N,K) FOR N = 16

K= 1		0	0	0	0	1
K= 2		0	0	0	0	32767
K= 3		0	0	0	0	7141686
K= 4		0	0	0	0	171798901
K= 5		0	0	0	0	1096190550
K= 6		0	0	0	0	2734926558
K= 7		0	0	0	0	3281882604
K= 8		0	0	0	0	2141764053
K= 9		0	0	0	0	820784250
K= 10		0	0	0	0	193754990
K= 11		0	0	0	0	28936908
K= 12		0	0	0	0	2757118
K= 13		0	0	0	0	165620
K= 14		0	0	0	0	6020
K= 15		0	0	0	0	120
K= 16		0	0	0	0	1

FOR N= 16 , MAX IS FOR K= 7

SIGMA =	0	0	0	0	0	10480142147
SUM OF FIRST 4 TERMS=	0	0	0	0	0	1275163905

S(N,K) FOR N = 17

K= 1		0	0	0	0	1
K= 2		0	0	0	0	65535
K= 3		0	0	0	0	21457825
K= 4		0	0	0	0	694337290
K= 5		0	0	0	0	5652751651
K= 6		0	0	0	0	17505749898
K= 7		0	0	0	0	25708104786
K= 8		0	0	0	0	20415995028
K= 9		0	0	0	0	9528822303
K= 10		0	0	0	0	2758334150
K= 11		0	0	0	0	512060978
K= 12		0	0	0	0	62022324
K= 13		0	0	0	0	4910178
K= 14		0	0	0	0	249900
K= 15		0	0	0	0	7820
K= 16		0	0	0	0	136
K= 17		0	0	0	0	1

FOR N= 17 , MAX IS FOR K= 7

SIGMA =	0	0	0	0	0	82864869804

S(N,K) FOR N = 18

K= 1		0	0	0	0	1
K= 2		0	0	0	0	131071
K= 3		0	0	0	0	64439010
K= 4		0	0	0	0	2798806985
K= 5		0	0	0	0	28958095545
K= 6		0	0	0	0	110687251039
K= 7		0	0	0	0	197462483400
K= 8		0	0	0	0	189036065010
K= 9		0	0	0	0	106175395755
K= 10		0	0	0	0	37112163803
K= 11		0	0	0	0	8391004908
K= 12		0	0	0	0	1256328866
K= 13		0	0	0	0	125854638
K= 14		0	0	0	0	8408778
K= 15		0	0	0	0	367200
K= 16		0	0	0	0	9996
K= 17		0	0	0	0	153
K= 18		0	0	0	0	1

FOR N= 18 , MAX IS FOR K= 7

SIGMA =	0	0	0	0	0	682076806159

S(N,K) FOR N = 19

K= 1	0	0	0	0	1
K= 2	0	0	0	0	262143
K= 3	0	0	0	0	193448101
K= 4	0	0	0	0	11259666950
K= 5	0	0	0		0147589284710
K= 6	0	0	0		0693081601779
K= 7	0	0	0		1492924634839
K= 8	0	0	0		1709751003480
K= 9	0	0	0		1144614626805
K= 10	0	0	0		0477297033785
K= 11	0	0	0		0129413217791
K= 12	0	0	0	0	23466951300
K= 13	0	0	0	0	2892439160
K= 14	0	0	0	0	243577530
K= 15	0	0	0	0	13916778
K= 16	0	0	0	0	527136
K= 17	0	0	0	0	12597
K= 18	0	0	0	0	171
K= 19	0	0	0	0	1

FOR N= 19 , MAX IS FOR K= 8

SIGMA =

0 0 0 0 5832742205057

S(N,K) FOR N = 20

K= 1	0	0	0	0	1
K= 2	0	0	0	0	524287
K= 3	0	0	0	0	580606446
K= 4	0	0	0	0	45232115901
K= 5	0	0	0		0749206090500
K= 6	0	0	0		4306078895384
K= 7	0	0	0		11143554045652
K= 8	0	0	0		15170932662679
K= 9	0	0	0	12	11282644725
K= 10	0	0	0		5917584964655
K= 11	0	0	0		1900842429486
K= 12	0	0	0		0411016633391
K= 13	0	0	0	0	61068660380
K= 14	0	0	0	0	6302524580
K= 15	0	0	0	0	452329200
K= 16	0	0	0	0	22350954
K= 17	0	0	0	0	741285
K= 18	0	0	0	0	15675
K= 19	0	0	0	0	190
K= 20	0	0	0	0	1

FOR N= 20 , MAX IS FOR K= 8

SIGMA =

0 0 0 0 51724158235372

S(N,K) FOR N = 21

K= 1	0	0	0	0	1
K= 2	0	0	0	0	1048575
K= 3	0	0	0	0	1742343625
K= 4	0	0	0		0181509070050
K= 5	0	0	0		3791262568401
K= 6	0	0	0		26585679462804
K= 7	0	0	0		82310957214948
K= 8	0	0	0		132511015347084
K= 9	0	0	0		123272476465204
K= 10	0	0	0		71187132291275
K= 11	0	0	0		26826851689001
K= 12	0	0	0		6833042030178
K= 13	0	0	0		1204909218331
K= 14	0	0	0		0149304004500
K= 15	0	0	0	0	13087462580
K= 16	0	0	0	0	809944464
K= 17	0	0	0	0	34952799
K= 18	0	0	0	0	1023435
K= 19	0	0	0	0	19285
K= 20	0	0	0	0	210
K= 21	0	0	0	0	1

FOR N= 21 , MAX IS FOR K= 8

SIGMA =

0 0 0 0 474869816156751

S(N,K) FOR N = 22

K= 1	0	0	0	0	1
K= 2	0	0	0	0	2097151
K= 3	0	0	0	0	5228079450
K= 4	0	0	0		0727778623825
K= 5	0	0	0		19137821912055
K= 6	0	0	0		163305339345225
K= 7	0	0	0		602762379967440
K= 8	0	0	0		1142399079991620
K= 9	0	0	0		1241963303533920
K= 10	0	0	0		835143799377954
K= 11	0	0	0		366282500870286
K= 12	0	0	0		108823356051137
K= 13	0	0	0		22496861868481
K= 14	0	0	0		3295165281331
K= 15	0	0	0		0345615943200
K= 16	0	0	0	0	26046574004
K= 17	0	0	0	0	1404142047
K= 18	0	0	0	0	53374629
K= 19	0	0	0	0	1389850
K= 20	0	0	0	0	23485
K= 21	0	0	0	0	231
K= 22	0	0	0	0	1

FOR N= 22 , MAX IS FOR K= 9

SIGMA =

0 0 0 0 4506715738447323

S(N,K) FOR N = 23

K= 1	0	0	0	0 1
K= 2	0	0	0	0 4194303
K= 3	0	0	0	0 15686335501
K= 4	0	0	0	2916342574750
K= 5	0	0	0	96416888184100
K= 6	0	0	0	998969857983405
K= 7	0	0	0	4382641999117305
K= 8	0	0	0	9741955019900400
K= 9	0	0	0	12320 68811796900
K= 10	0	0	0	9593401297313460
K= 11	0	0	0	4864251308951100
K= 12	0	0	0	1672162773483930
K= 13	0	0	0	401282560341390
K= 14	0	0	0	68629175807115
K= 15	0	0	0	8479404429331
K= 16	0	0	0	0762361127264
K= 17	0	0	0	0 49916988803
K= 18	0	0	0	0 2364885369
K= 19	0	0	0	0 79781779
K= 20	0	0	0	0 1859550
K= 21	0	0	0	0 28336
K= 22	0	0	0	0 253
K= 23	0	0	0	0 1

FOR N= 23 , MAX IS FOR K= 9

SIGMA =

0	0	0	0	44152 5855084346

S(N,K) FOR N = 24

K= 1	0	0	0	0 1
K= 2	0	0	0	0 8388607
K= 3	0	0	0	0 47063200806
K= 4	0	0	0	11681056634501
K= 5	0	0	0	485 783495250
K= 6	0	0	0	6090236036084530
K= 7	0	0	0	31677463851804540
K= 8	0	0	0	82318282158320505
K= 9	0	0	0	120622574326072500
K= 10	0	0	0	108254 81784931500
K= 11	0	0	0	63100165695775560
K= 12	0	0	0	24930204590758260
K= 13	0	0	0	6888836057922000
K= 14	0	0	0	1362 91021641000
K= 15	0	0	0	195820242247080
K= 16	0	0	0	20677182465555
K= 17	0	0	0	1610949936915
K= 18	0	0	0	0 92484925445
K= 19	0	0	0	0 3880739170
K= 20	0	0	0	0 116972779
K= 21	0	0	0	0 2454606
K= 22	0	0	0	0 33902
K= 23	0	0	0	0 276
K= 24	0	0	0	0 1

FOR N= 24 , MAX IS FOR K= 9

SIGMA =

0	0	0	0	445958869294805289

S(N,K) FOR N = 25

K= 1	0	0	0	0 1
K= 2	0	0	0	0 16777215
K= 3	0	0	0	0141197991025
K= 4	0	0	0	46771289738810
K= 5	0	0	0	2436684974110751
K= 6	0	0	0	37026417000002430
K= 7	0	0	0	227832482998716310
K= 8	0	0	0	690223721118368580
K= 9	0	0	0	1167921451092973005
K= 10	0	0	0	1203163392175387500
K= 11	0	0	0	802355904438462660
K= 12	0	0	0	362262620784874680
K= 13	0	0	0	114485 73343744260
K= 14	0	0	0	25958110360896000
K= 15	0	0	0	4299394655347200
K= 16	0	0	0	526655161695960
K= 17	0	0	0	48 63331393110
K= 18	0	0	0	3275678594925
K= 19	0	0	0	0166218969675
K= 20	0	0	0	0 6220194750
K= 21	0	0	0	0 168519505
K= 22	0	0	0	0 3200450
K= 23	0	0	0	0 40250
K= 24	0	0	0	0 300
K= 25	0	0	0	0 1

FOR N= 25 , MAX IS FOR K= 10

SIGMA =

0	0	0	0	4638590332229999353

SUM OF FIRST 5 TERMS=

0	0	0	0	39510 14478620232

S(N,K) FOR N = 26

K= 1	0	0	0	0 1
K= 2	0	0	0	0 33554431
K= 3	0	0	0	0423610750290
K= 4	0	0	0	187226356946265
K= 5	0	0	0	12230196160292565
K= 6	0	0	0	224595186974125331
K= 7	0	0	0	1631853797991016600
K= 8	0	0	0	5749622251945664950
K= 9	0	0	0	11201516780955125625
K= 10	0	0	0	13199555372846848005
K= 11	0	0	0	10029078340998476760
K= 12	0	0	0	5149507353856958820
K= 13	0	0	0	1850568574253550060
K= 14	0	0	0	477898618396288260
K= 15	0	0	0	90449 30191104000
K= 16	0	0	0	12725877242482560
K= 17	0	0	0	1343731795378830
K= 18	0	0	0	107 25546101760
K= 19	0	0	0	6433839018750
K= 20	0	0	0	0290622864675
K= 21	0	0	0	0 9759104355
K= 22	0	0	0	0 238929405
K= 23	0	0	0	0 4126200
K= 24	0	0	0	0 47450
K= 25	0	0	0	0 325
K= 26	0	0	0	0 1

FOR N= 26 , MAX IS FOR K= 10

SIGMA =

0	0	0	0	4963124652361875627 4

S(N,K) FOR N = 27

K= 1	0	0	0	0	1
K= 2	0	0	0	0	67108863
K= 3	0	0	0		1270865805301
K= 4	0	0	0		749329038535350
K= 5	0	0	0		61338207158409090
K= 6	0	0	0		1359801318005044551
K= 7	0	0	0		11647571772911241531
K= 8	0	0	0		47628831813556336200
K= 9	0	0	0		106563273280541795575
K= 10	0	0	0		143197070509423605675
K= 11	0	0	0		123519417123830092365
K= 12	0	0	0		71823166587281982600
K= 13	0	0	0		29206898819153109600
K= 14	0	0	0		8541149231801585700
K= 15	0	0	0		1834634 71262848260
K= 16	0	0	0		294063 66070824960
K= 17	0	0	0		35569317763922670
K= 18	0	0	0		3270191625210510
K= 19	0	0	0		229268487458010
K= 20	0	0	0		12246296312250
K= 21	0	0	0		0495564056130
K= 22	0	0	0	0	15015551265
K= 23	0	0	0	0	333832005
K= 24	0	0	0	0	5265000
K= 25	0	0	0	0	55575
K= 26	0	0	0	0	351
K= 27	0	0	0	0	1

FOR N= 27 , MAX IS FOR K= 10
SIGMA =

0	0	0	0	545717047936059989389

S(N,K) FOR N = 28

K= 1	0	0	0	0	1
K= 2	0	0	0	0	134217727
K= 3	0	0	0		3812664524766
K= 4	0	0	0		2998587019946701
K= 5	0	0	0		307440364830580800
K= 6	0	0	0		8220146115188676396
K= 7	0	0	0		82892803728383735268
K= 8	0	0	0		392678226281361931131
K= 9	0	0	0		1006698291338432496375
K= 10	0	0	0		1538533978374777852325
K= 11	0	0	0		1501910658871554621690
K= 12	0	0	0		985397416171213883565
K= 13	0	0	0		451512851236272407400
K= 14	0	0	0		148782988 64375309400
K= 15	0	0	0		36060660300744309600
K= 16	0	0	0		6539643128396047620
K= 17	0	0	0		898741468057510350
K= 18	0	0	0		94432767017711850
K= 19	0	0	0		7626292886912700
K= 20	0	0	0		474194413703010
K= 21	0	0	0		22653141490980
K= 22	0	0	0		0825906183960
K= 23	0	0	0	0	22693687380
K= 24	0	0	0	0	460192005
K= 25	0	0	0	0	6654375
K= 26	0	0	0	0	64701
K= 27	0	0	0	0	378
K= 28	0	0	0	0	1

FOR N= 28 , MAX IS FOR K= 10
SIGMA =

0	0	0	0	6160539404599934652455

S(N,K) FOR N = 29

K= 1	0	0	0	0	1
K= 2	0	0	0	0	268435455
K= 3	0	0	0		11438127792025
K= 4	0	0	0		11998160744311570
K= 5	0	0	0		1540200411172850701
K= 6	0	0	0		49628317 55962639176
K= 7	0	0	0		588469772213874823272
K= 8	0	0	0		3224318613979279184316
K= 9	0	0	0		9452962848327254398506
K= 10	0	0	0	0	16392038075 86211019625
K= 11	0	0	0	0	18059551225961878690915
K= 12	0	0	0	0	13326679652926121224470
K= 13	0	0	0		6855064482242755179765
K= 14	0	0	0		2534474684137526739000
K= 15	0	0	0		689692892575539953400
K= 16	0	0	0		140694950355081071520
K= 17	0	0	0		21818248 85373723570
K= 18	0	0	0		2598531274376323650
K= 19	0	0	0		239332331869053150
K= 20	0	0	0		17110181160972900
K= 21	0	0	0		949910385013590
K= 22	0	0	0		40823077538100
K= 23	0	0	0		1347860993700
K= 24	0	0	0	0	33738295500
K= 25	0	0	0	0	626551380
K= 26	0	0	0	0	8336601
K= 27	0	0	0	0	74907
K= 28	0	0	0	0	406
K= 29	0	0	0	0	1

FOR N= 29 , MAX IS FOR K= 11
SIGMA =

0	0	0	0	71339801938860275191172

S(N,K) FOR N = 30

K= 1	0	0	0	0	1
K= 2	0	0	0	0	536870911
K= 3	0	0	0		34314651811530
K= 4	0	0	0		48004 81105038305
K= 5	0	0	0		7713000216608565075
K= 6	0	0	0		299310102746948685757
K= 7	0	0	0		4168916722553086402080
K= 8	0	0	0	0	26383018684 48108297800
K= 9	0	0	0	0	88300984248924568770870
K= 10	0	0			0173373343599189364594756
K= 11	0	0			0215047101560666876619690
K= 12	0	0			0177979707061 75333384555
K= 13	0	0			0102442517922 81938561415
K= 14	0	0	0		42337710060168129525765
K= 15	0	0	0		12879868072770626040000
K= 16	0	0	0		2940812098256837097720
K= 17	0	0	0		511605167806434372210
K= 18	0	0	0		68591811 24147549270
K= 19	0	0	0		7145845579888333500
K= 20	0	0	0		581535955088511150
K= 21	0	0	0		37058299246258290
K= 22	0	0	0		1848 18090851790
K= 23	0	0	0		71823880393200
K= 24	0	0	0		2157580085700
K= 25	0	0	0	0	49402080000
K= 26	0	0	0	0	843303006
K= 27	0	0	0	0	10359090
K= 28	0	0	0	0	86275
K= 29	0	0	0	0	435
K= 30	0	0	0	0	1

FOR N= 30 , MAX IS FOR K= 11
SIGMA =

0	0	0	0846749014511809332450147

Distributions

Partitions and Coalitions are special cases of distributions. In distributions we ask for the number of ways in which n(like, unlike) objects can be distributed over k (like, unlike) cells (groups, compartments, etc.). We shall denote the four kinds of distribution that arise from the distinguishability or indistinguishability of objects or cells by subscripts "o" or "c", where the presence of subscripts "o" or "c" indicates distinguishability of objects or cells respectively. Then we have:

1. Like objects, like cells:

$$N(n,k) = p(n,k)$$

2. Unlike objects, like cells:

$$N_o(n,k) = S(n,k)$$

3. Like objects, unlike cells:

$$N_c(n,k) = \binom{n-1}{k-1}$$

4. Unlike objects, unlike cells:

$$N_{o,c}(N_1k) = k!\ S(n,k)$$

where k! represents the factorial expansion k, i.e., k(k-1)(k-2)...(k-(k-1)); S(n,k) the Stirling number of the Second kind; p(n,k) the number of partitions of n into exactly k parts; and the expression

$$\binom{p}{q} = \frac{p!}{q!\,(p-q)!}$$

the q-th binomial coefficient in the expansion of $(a+b)^p$.

The following table lists the outcome of these various distinctions for numbers of elements from 1→6.

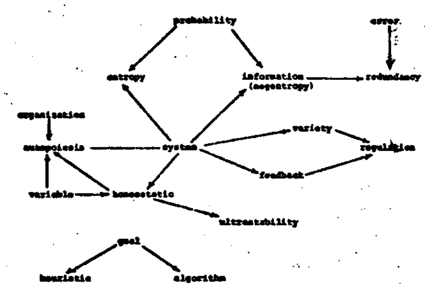
TABLE OF DISTRIBUTIONS

$N_{..}(n,k)$	k ↓	n → 1	2	3	4	5	6
$N_{o,c}(n,1)$		1	1	1	1	1	1
$N_o(n,1)$	1	1	1	1	1	1	1
$N_c(n,1)$		1	1	1	1	1	1
$N(n,1)$		1	1	1	1	1	1
$N_{o,c}(n,2)$			2	6	14	30	62
$N_o(n,2)$	2		1	3	7	15	31
$N_c(n,2)$			1	2	3	4	5
$N(n,2)$			1	1	2	2	3
$N_{o,c}(n,3)$				6	36	150	540
$N_o(n,3)$	3			1	6	25	90
$N_c(n,3)$				1	3	6	10
$N(n,3)$				1	1	2	3
$N_{o,c}(n,4)$					24	240	1560
$N_o(n,4)$	4				1	10	65
$N_c(n,4)$					1	4	10
$N(n,4)$					1	1	2
$N_{o,c}(n,5)$						120	1800
$N_o(n,5)$	5					1	15
$N_c(n,5)$						1	5
$N(n,5)$						1	1
$N_{o,c}(n,6)$							720
$N_o(n,6)$	6						1
$N_c(n,6)$							1
$N(n,6)$							1

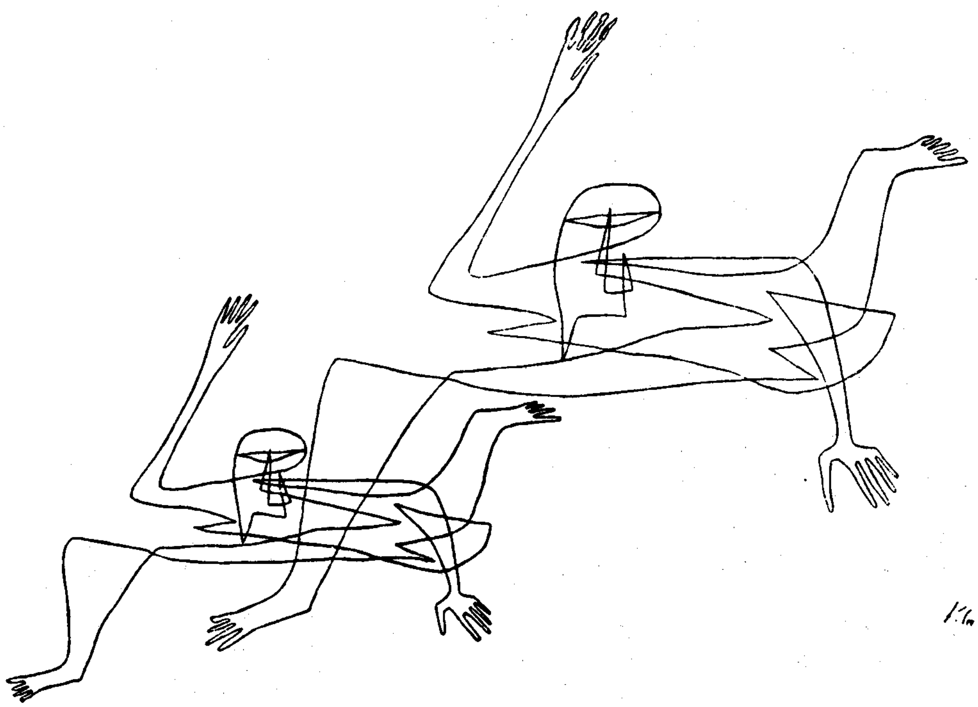
REDUNDANCY

(1) In information theory, the enrichment of information in a message to protect it from degradation by noise.

(2) In cybernetics, usage (1) is familiar, but the term is also applied to extra channels in a network that are intended to guard a whole system against the failure of an entire channel. It is possible to calculate mathematically how much redundancy is required to reduce the risk of a mistake (getting the message wrong in (1), or failure of the system in (2)) to an arbitrarily small degree.



[S.B.]



ON SELF-ORGANIZING SYSTEMS AND THEIR ENVIRONMENTS*

H. von FOERSTER
Department of Electrical Engineering
University of Illinois, Urbana, Illinois

I AM somewhat hesitant to make the introductory remarks of my presentation, because I am afraid I may hurt the feelings of those who so generously sponsored this conference on self-organizing systems. On the other hand, I believe, I may have a suggestion on how to answer Dr. Weyl's question which he asked in his pertinent and thought-provoking introduction: "What makes a self-organizing system?" Thus, I hope you will forgive me if I open my paper by presenting the following thesis: "There are no such things as self-organizing systems!"

In the face of the title of this conference I have to give a rather strong proof of this thesis, a task which may not be at all too difficult, if there is not a secret purpose behind this meeting to promote a conspiracy to dispose of the Second Law of Thermodynamics. I shall now prove the non-existence of self-organizing systems by *reductio ad absurdum* of the assumption that there is such a thing as a self-organizing system.

Assume a finite universe, U_0, as small or as large as you wish (see Fig. 1*a*), which is enclosed in an adiabatic shell which separates this finite universe from any "meta-universe" in which it may be immersed. Assume, furthermore, that in this universe, U_0, there is a closed surface which divides this universe into two mutually exclusive parts: the one part is completely occupied with a self-organizing system S_0, while the other part we may call the environment E_0 of this self-organizing system: S_0 & $E_0 = U_0$.

I may add that it is irrelevant whether we have our self-organizing system inside or outside the closed surface. However, in Fig. 1 the

* Supported by the Information Systems Branch of the Office of Naval Research under Contract Nonr. 1834 (21).

system is assumed to occupy the interior of the dividing surface.

Undoubtedly, if this self-organizing system is permitted to do its job of organizing itself for a little while, its entropy must have decreased during this time:

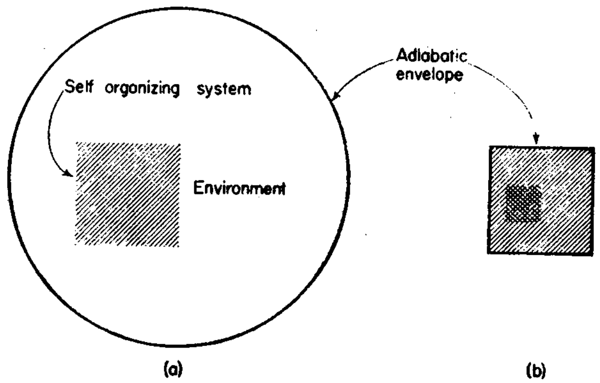


FIG. 1.

$$\frac{\delta S_s}{\delta t} < 0,$$

otherwise we would not call it a self-organizing system, but just a mechanical $\delta S_s/\delta t = 0$, or a thermodynamical $\delta S_s/\delta t > 0$ system. In order to accomplish this, the entropy in the remaining part of our finite universe, i.e. the entropy in the environment must have increased

$$\frac{\delta S_E}{\delta t} > 0,$$

otherwise the Second Law of Thermodynamics is violated. If now some of the processes which contributed to the decrease of entropy of the system are irreversible we will find the entropy of the universe U_0 at a higher level than before our system started to organize itself, hence the state of the universe will be more disorganized than before $\delta S_U/\delta t > 0$, in other words, the activity of the system was a disorganizing one, and we may justly call such a system a " disorganizing system."

However, it may be argued that it is unfair to the system to make it responsible for changes in the whole universe and that this apparent inconsistency came about by not only paying attention to the system proper but also including into the consideration the environment of the system. By drawing too large an adiabatic envelope one may include processes not at all relevant to this argument. All right then, let us have the adiabatic envelope coincide with the closed surface which previously separated the system from its environment (Fig. 1*b*). This step will not only invalidate the above argument, but will also enable me to show that if one assumes that this envelope contains the self-organizing system proper, this system turns out to be not only just a disorganizing system but even a self-disorganizing system.

It is clear from my previous example with the large envelope, that here too—if irreversible processes should occur—the entropy of the system now within the envelope must increase, hence, as time goes on, the system would disorganize itself, although in certain regions the entropy may indeed have decreased. One may now insist that we should have wrapped our envelope just around this region, since it appears to be the proper self-organizing part of our system. But again, I could employ that same argument as before, only to a smaller region, and so we could go on for ever, until our would-be self-organizing system has vanished into the eternal happy hunting grounds of the infinitesimal.

In spite of this suggested proof of the non-existence of self-organizing systems, I propose to continue the use of the term " self-organizing system," whilst being aware of the fact that this term becomes meaningless, unless the system is in close contact with an environment, *which posseses available energy and order*, and with which our system is in a state of perpetual interaction, such that it somehow manages to " live " on the expenses of this environment.

Although I shall not go into the details of the interesting discussion of the energy flow from the environment into the system and out again, I may briefly mention the two different schools of thought associated with this problem, namely, the one which considers energy flow and signal flow as a strongly linked, single-channel affair (i.e. the message carries also the food, or, if you wish, signal and food are synonymous) while the other viewpoint carefully separates these two, although there exists in this theory a significant interdependence between signal flow and energy availability.

I confess that I do belong to the latter school of thought and I am particularly happy that later in this meeting Mr. Pask, in his paper *The Natural History of Networks*,[2] will make this point of view much clearer than I will ever be able to do.

What interests me particularly at this moment is not so much the energy from the environment which is digested by the system, but its utilization of environmental order. In other words, the question I would like to answer is: " How much order can our system assimilate from its environment, if any at all? "

Before tackling this question, I have to take two more hurdles, both of which represent problems concerned with the environment. Since you have undoubtedly observed that in my philosophy about self-organizing systems the environment of such systems is a *conditio sine qua non* I am first of all obliged to show in which sense we may talk about the existence of such an environment. Second, I have to show that, if there exists such an environment, it must possess structure.

The first problem I am going to eliminate is perhaps one of the oldest philosophical problems with which mankind has had to live. This problem arises when we, men, consider ourselves to be self-organizing systems. We may insist that introspection does not permit us to decide whether the world as we see it is " real," or just a phantasmagory, a dream, an illusion of our fancy. A decision in this dilemma is in so far pertinent to my discussion, since—if the latter alternative should hold true—my original thesis asserting the nonsensicality of the conception of an isolated self-organizing system would pitiably collapse.

I shall now proceed to show the reality of the world as we see it, by *reductio ad absurdum* of the thesis: this world is only in our imagination and the only reality is the imagining " I ".

Thanks to the artistic assistance of Mr. Pask who so beautifully illustrated this and some of my later assertions,* it will be easy for me to develop my argument.

Assume for the moment that I am the successful business man with the bowler hat in Fig. 2, and I insist that I am the sole reality,

*Figures 2, 5 and 6

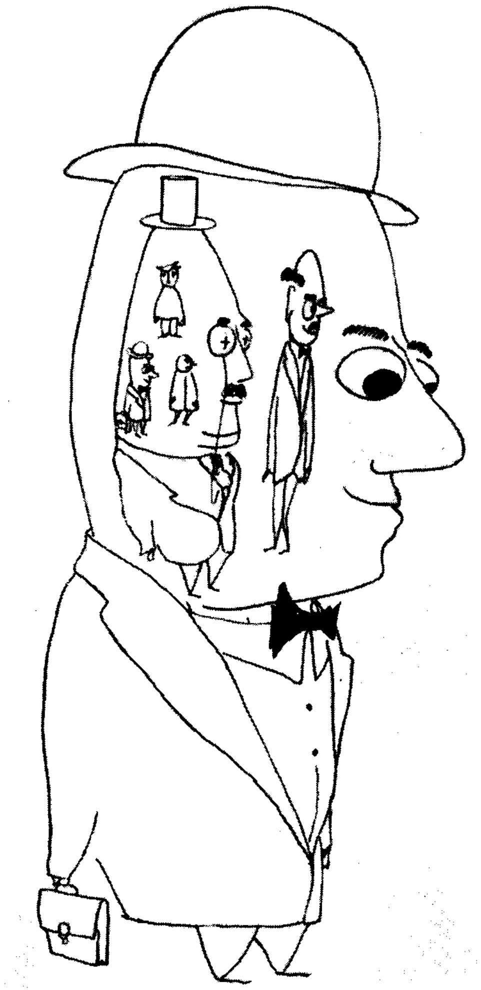


FIG. 2.

while everything else appears only in my imagination. I cannot deny that in my imagination there will appear people, scientists, other successful businessmen, etc., as for instance in this conference. Since I find these apparitions in many respects similar to myself, I have to grant them the privilege that they themselves may insist that they are the sole reality and everything else is only a concoction of their imagination. On the other hand, they cannot deny that their fantasies will be populated by people—and one of them may be I, with bowler hat and everything!

With this we have closed the circle of our contradiction: If I assume that I am the sole reality, it turns out that I am the imagination of somebody else, who in turn assumes that *he* is the sole reality. Of course, this paradox is easily resolved, by postulating the reality of the world in which we happily thrive.

Having re-established reality, it may be interesting to note that reality appears as a consistent reference frame for at least two observers. This becomes particularly transparent, if it is realized that my "proof" was exactly modeled after the "Principle of Relativity," which roughly states that, if a hypothesis which is applicable to a set of objects holds for one object and it holds for another object, then it holds for both objects simultaneously, the hypothesis is acceptable for all objects of the set. Written in terms of symbolic logic, we have:

$$(Ex)\,[H(a)\ \&\ H(x) \rightarrow H(a+x)] \rightarrow (x)\,H(x) \qquad (1)$$

Copernicus could have used this argument to his advantage, by pointing out that if we insist on a geocentric system, $[H(a)]$, the Venusians, e.g. could insist on a venucentric system $[(Hx)]$. But since we cannot be both, center and epicycloid at the same time $[H(a+x)]$, something must be wrong with a planetocentric system.

However, one should not overlook that the above expression, $\mathscr{R}(H)$ is not a tautology, hence it must be a meaningful statement.* What it does, is to establish a way in which we may talk about the existence of an environment.

* This was observed by Wittgenstein,(6) although he applied this consideration to the principle of mathematical induction. However, the close relation between the induction and the relativity principle seems to be quite evident. I would even venture to say that the principle of mathematical induction is the relativity principle in number theory.

Before I can return to my original question of how much order a self-organizing system may assimilate from its environment, I have to show that there is some structure in our environment. This can be done very easily indeed, by pointing out that we are obviously not yet in the dreadful state of Boltzmann's "Heat-Death." Hence, presently still the entropy increases, which means that there must be some order—at least now—otherwise we could not lose it.

Let me briefly summarize the points I have made until now:

(1) By a self-organizing system I mean that part of a system that eats energy and order from its environment.

(2) There is a reality of the environment in a sense suggested by the acceptance of the principle of relativity.

(3) The environment has structure.

Let us now turn to our self-organizing systems. What we expect is that the systems are increasing their internal order. In order to describe this process, first, it would be nice if we would be able to define what we mean by "internal," and second, if we would have some measure of order.

The first problem arises whenever we have to deal with systems which do not come wrapped in a skin. In such cases, it is up to us to define the closed boundary of our system. But this may cause some trouble, because, if we specify a certain region in space as being intuitively the proper place to look for our self-organizing system, it may turn out that this region does not show self-organizing properties at all, and we are forced to make another choice, hoping for more luck this time. It is this kind of difficulty which is encountered, e.g., in connection with the problem of the "localization of functions" in the cerebral cortex.

Of course, we may turn the argument the other way around by saying that we define our boundary at any instant of time as being the envelope of that region in space which shows the desired increase in order. But here we run into some trouble again; because I do not know of any gadget which would indicate whether it is plugged into a self-*dis*organizing or self-organizing region, thus providing us with a sound operational definition.

Another difficulty may arise from the possibility that these self-organizing regions may not only constantly move in space and change in shape, they may appear and disappear spontaneously here and there, requiring the "ordometer" not only to follow these all-elusive systems, but also to sense the location of their formation.

With this little digression I only wanted to point out that we have to be very cautious in applying the word "inside" in this context, because, even if the position of the observer has been stated, he may have a tough time saying what he sees.

Let us now turn to the other point I mentioned before, namely, trying to find an adequate measure of order. It is my personal feeling that we wish to describe by this term two states of affairs. First, we may wish to account for apparent relationships between elements of a set which would impose some constraints as to the possible arrangements of the elements of this system. As the organization of the system grows, more and more of these relations should become apparent. Second, it seems to me that order has a relative connotation, rather than an absolute one, namely, with respect to the maximum disorder the elements of the set may be able to display. This suggests that it would be convenient if the measure of order would assume values between zero and unity, accounting in the first case for maximum disorder and, in the second case, for maximum order. This eliminates the choice of "neg-entropy" for a measure of order, because neg-entropy always assumes finite values for systems being in complete disorder. However, what Shannon[3] has defined as "redundancy" seems to be tailor-made for describing order as I like to think of it. Using Shannon's definition for redundancy we have:

$$R = 1 - \frac{H}{H_m} \qquad (2)$$

whereby H/H_m is the ratio of the entropy H of an information source to the maximum value, H_m, it could have while still restricted to the same symbols. Shannon calls this ratio the "relative entropy." Clearly, this expression fulfills the requirements for a measure of order as I have listed them before. If the system is in its maximum disorder $H = H_m$, R becomes zero; while, if the elements of the system are arranged such that, given one element, the position of all other elements are determined, the entropy—or the degree of uncertainty—vanishes, and R becomes unity, indicating perfect order.

What we expect from a self-organizing system is, of course, that, given some initial value of order in the system, this order is going to increase as time goes on. With our expression (2) we can at once state the criterion for a system to be self-organizing, namely, that the rate of change of R should be positive:

$$\frac{\delta R}{\delta t} > 0 \qquad (3)$$

Differentiating eq. (2) with respect to time and using the inequality (3) we have:

$$\frac{\delta R}{\delta t} = -\frac{H_m(\delta H/\delta t) - H(\delta H_m/\delta t)}{H_m^2} \qquad (4)$$

Since $H_m^2 > 0$, under all conditions (unless we start out with systems which can only be thought of as being always in perfect order: $H_m = 0$), we find the condition for a system to be self-organizing expressed in terms of entropies:

$$H \frac{\delta H_m}{\delta t} > H_m \frac{\delta H}{\delta t} \qquad (5)$$

In order to see the significance of this equation let me first briefly discuss two special cases, namely those, where in each case one of the two terms H, H_m is assumed to remain constant.

(a) $H_m = \text{const.}$

Let us first consider the case, where H_m, the maximum possible entropy of the system remains constant, because it is the case which is usually visualized when we talk about self-organizing systems. If H_m is supposed to be constant the time derivative of H_m vanishes, and we have from eq. (5):

$$\text{for} \quad \frac{\delta H_m}{\delta t} = 0 \ldots\ldots \frac{\delta H}{\delta t} < 0 \qquad (6)$$

This equation simply says that, when time goes on, the entropy of the system should decrease. We knew this already—but now we may ask, how can this be accomplished? Since the entropy of the system is dependent upon the probability distribution of the elements to be found in certain distinguishable states, it is clear that this probability distribution must change such that H is reduced. We may visualize this, and how this can be accomplished, by paying attention to the factors which determine the probability distribution. One of these factors could be that our elements possess certain properties which would make it more or less likely that an element is found to be in a certain state. Assume, for instance, the state under consideration is " to be in a hole of a certain size." The probability of elements with sizes larger than the hole to be found in this state is clearly zero. Hence, if the elements are slowly blown up like little balloons, the probability distribution will constantly change. Another factor influencing the probability distribution could be that our elements possess some other properties which determine the conditional probabilities of an element to be found in certain states, given the state of other elements in this system. Again, a change in these conditional probabilities will change the probability distribution, hence the entropy of the system. Since all these changes take place internally I'm going to make an " internal demon " responsible for these changes. He is the one, e.g. being busy blowing up the little balloons and thus changing the probability distribution, or shifting conditional probabilities by establishing ties between elements such that H is going to decrease. Since we have some familiarity with the task of this demon, I shall leave him for a moment and turn now to another one, by discussing the second special case I mentioned before, namely, where H is supposed to remain constant.

(*b*) $H = \text{const.}$

If the entropy of the system is supposed to remain constant, its time derivative will vanish and we will have from eq. (5)

$$\text{for} \qquad \frac{\delta H}{\delta t} = 0 \ldots\ldots \frac{\delta H_m}{\delta t} > 0 \qquad (7)$$

Thus, we obtain the peculiar result that, according to our previous definition of order, we may have a self-organizing system before us, if its possible maximum disorder is increasing. At first glance, it seems that to achieve this may turn out to be a rather trivial affair, because one can easily imagine simple processes where this condition is fulfilled. Take as a simple example a system composed of N elements which are capable of assuming certain observable states. In most cases a probability distribution for the number of elements in these states can be worked out such that H is maximized and an expression for H_m is obtained. Due to the fact that entropy (or, amount of information) is linked with the logarithm of the probabilities, it is not too difficult to show that expressions for H_m usually follow the general form*:

$$H_m = C_1 + C_2 \log_2 N.$$

This suggests immediately a way of increasing H_m, namely, by just increasing the number of elements constituting the system; in other words a system that grows by incorporating new elements will increase its maximum entropy and, since this fulfills the criterion for a system to be self-organizing (eq. 7), we must, by all fairness, recognize this system as a member of the distinguished family of self-organizing systems.

It may be argued that if just adding elements to a system makes this a self-organizing system, pouring sand into a bucket would make the bucket a self-organizing system. Somehow—to put it mildly—this does not seem to comply with our intuitive esteem for members of our distinguished family. And rightly so, because this argument ignores the premise under which this statement was derived, namely, that during the process of adding new elements to the system the entropy H of the system is to be kept constant. In the case of the bucket full of sand, this might be a ticklish task, which may conceivably be accomplished, e.g. by placing the newly admitted particles precisely in the same order with respect to some distinguishable states, say position, direction, etc. as those present at the instant of admission of the newcomers. Clearly, this task of increasing H_m by keeping H constant asks for superhuman skills and thus we may employ another demon whom I shall call the " external demon," and whose business it is to admit to the system only those elements, the state of which complies with the conditions of, at least, constant internal entropy. As you certainly have noticed, this demon is a close relative of Maxwell's demon, only that to-day these fellows don't come as good as they used to come, because before 1927[4] they could watch an arbitrary small hole through which the newcomer had to pass and could test with arbitrary high accuracy his momentum. To-day, however, demons watching

* See also Appendix.

closely a given hole would be unable to make a reliable momentum test, and vice versa. They are, alas, restricted by Heisenberg's uncertainty principle.

Having discussed the two special cases where in each case only one demon is at work while the other one is chained, I shall now briefly describe the general situation where both demons are free to move, thus turning to our general eq. (5) which expressed the criterion for a system to be self-organizing in terms of the two entropies H and H_m. For convenience this equation may be repeated here, indicating at the same time the assignments for the two demons D_i and D_e:

$$\underbrace{H}_{\text{Internal demon's results}} \times \underbrace{\frac{\delta H_m}{\delta t}}_{\text{External demon's efforts}} > \underbrace{H_m}_{\text{External demon's results}} \times \underbrace{\frac{\delta H}{\delta t}}_{\text{Internal demon's efforts}} \qquad (5)$$

From this equation we can now easily see that, if the two demons are permitted to work together, they will have a disproportionately easier life compared to when they were forced to work alone. First, it is not necessary that D_i is always decreasing the instantaneous entropy H, or D_e is always increasing the maximum possible entropy H_m; it is only necessary that the product of D_i's results with D_e's efforts is larger than the product of D_e's results with D_i's efforts. Second, if either H or H_m is large, D_e or D_i respectively can take it easy, because their efforts will be multiplied by the appropriate factors. This shows, in a relevant way, the interdependence of these demons. Because, if D_i was very busy in building up a large H, D_e can afford to be lazy, because his efforts will be multiplied by D_i's results, and vice versa. On the other hand, if D_e remains lazy too long, D_i will have nothing to build on and his output will diminish, forcing D_e to resume his activity lest the system ceases to be a self-organizing system.

In addition to this entropic coupling of the two demons, there is also an energetic interaction between the two which is caused by the energy requirements of the internal demon who is supposed to accomplish the shifts in the probability distribution of the elements comprising the system. This requires some energy, as we may remember from our previous example, where somebody has to blow up the little balloons. Since this energy has been taken from the environment, it will affect the activities of the external demon who may be confronted with a problem when he attempts to supply the system with choice-entropy he must gather from an energetically depleted environment.

In concluding the brief exposition of my demonology, a simple diagram may illustrate the double linkage between the internal and the external demon which makes them entropically (H) and energetically (E) interdependent.

For anyone who wants to approach this subject from the point of view of a physicist, and who is conditioned to think in terms of thermodynamics and statistical mechanics, it is impossible not to refer to the beautiful little monograph by Erwin Schrodinger *What is Life.*[5] Those of you who are familiar with this book may remember that Schrodinger admires particularly two remarkable features of living organisms. One is the incredible high order of the genes, the " hereditary code-scripts " as he calls them, and the other one is the marvelous stability of these organized units whose delicate structures remain almost untouched despite their exposure to thermal agitation by being immersed—e.g. in the case of mammals —into a thermostat, set to about 310°K.

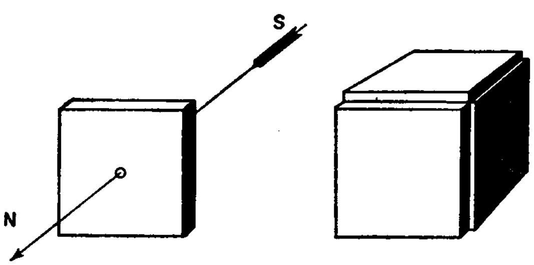
In the course of his absorbing discussion, Schrodinger draws our attention to two different basic " mechanisms " by which orderly events can be produced: " The statistical mechanism which produces order from disorder and the ... [other] one producing ' order from order '."

While the former mechanism, the " order from disorder " principle is merely referring to " statistical laws " or, as Schrodinger puts it, to " the magnificent order of exact physical law coming forth from atomic and molecular disorder," the latter mechanism, the " order from order " principle is, again in his words: " the real clue to the understanding of life." Already earlier in his book Schrodinger develops this principle very clearly and states: " What an organism feeds upon is negative entropy." I think my demons would agree with this, and I do too.

However, by reading recently through Schrodinger's booklet I wondered how it could happen that his keen eyes escaped what I would consider a " second clue " to the understanding of life, or—if it is fair to say—of self-organizing systems. Although the principle I have in mind may, at first glance, be mistaken for Schrodinger's " order from disorder " principle, it has in fact nothing in common with it. Hence, in order to stress the difference between the two, I shall call the principle I am going to introduce to you presently the " order from noise " principle. Thus, in my restaurant self-organizing systems do not only feed upon order, they will also find noise on the menu.

Let me briefly explain what I mean by saying that a self-organizing system feeds upon noise by using an almost trivial, but nevertheless amusing example.

Assume I get myself a large sheet of permanent magnetic material which is strongly magnetized perpendicular to the surface, and I cut from this sheet a large number of little squares (Fig. 3*a*). These



(a) (b)

FIG. 3. (*a*) Magnetized square.
(*b*) Cube, family I.

little squares I glue to all the surfaces of small cubes made of light, unmagnetic material, having the same size as my squares (Fig. 3*b*). Depending upon the choice of which sides of the cubes have the magnetic north pole pointing to the outside (Family I), one can produce precisely ten different families of cubes as indicated in Fig. 4.

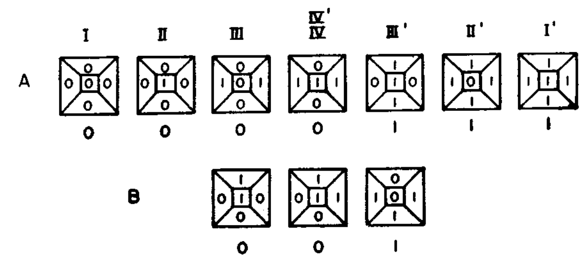


FIG. 4. Ten different families of cubes (see text).

Suppose now I take a large number of cubes, say, of family I, which is characterized by all sides having north poles pointing to the outside (or family I′ with all south poles), put them into a large box which is also filled with tiny glass pebbles in order to make these cubes float under friction and start shaking this box. Certainly, nothing very striking is going to happen: since the cubes are all repelling each other, they will tend to distribute themselves in the available space such that none of them will come too close to its fellow-cube. If, by putting the cubes into the box, no particular ordering principle was observed, the entropy of the system will remain constant, or, at worst, increase a small amount.

In order to make this game a little more amusing, suppose now I collect a population of cubes where only half of the elements are again members belonging to family I (or I′) while the other half are members of family II (or II′) which is characterized by having only one side of different magnetism pointing to the outside. If this population is put into my box and I go on shaking, clearly, those cubes with the single different pole pointing to the outside will tend, with overwhelming probability, to mate with members of the other family, until my cubes have almost all paired up. Since the conditional probabilities of finding a member of family II, given the locus of a member of family I, has very much increased, the entropy of the system has gone down, hence we have more order after the shaking than before. It is easy to show* that in this case

* See Appendix.

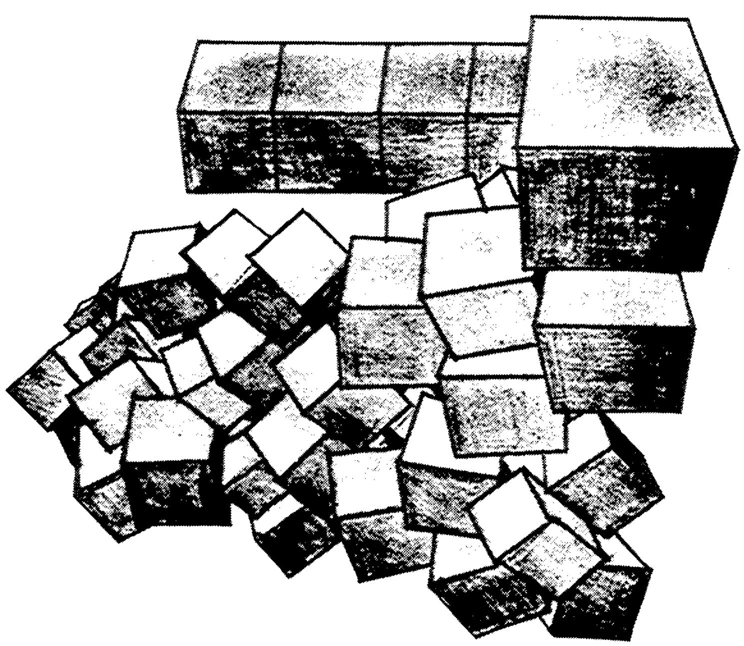


FIG. 5. Before.

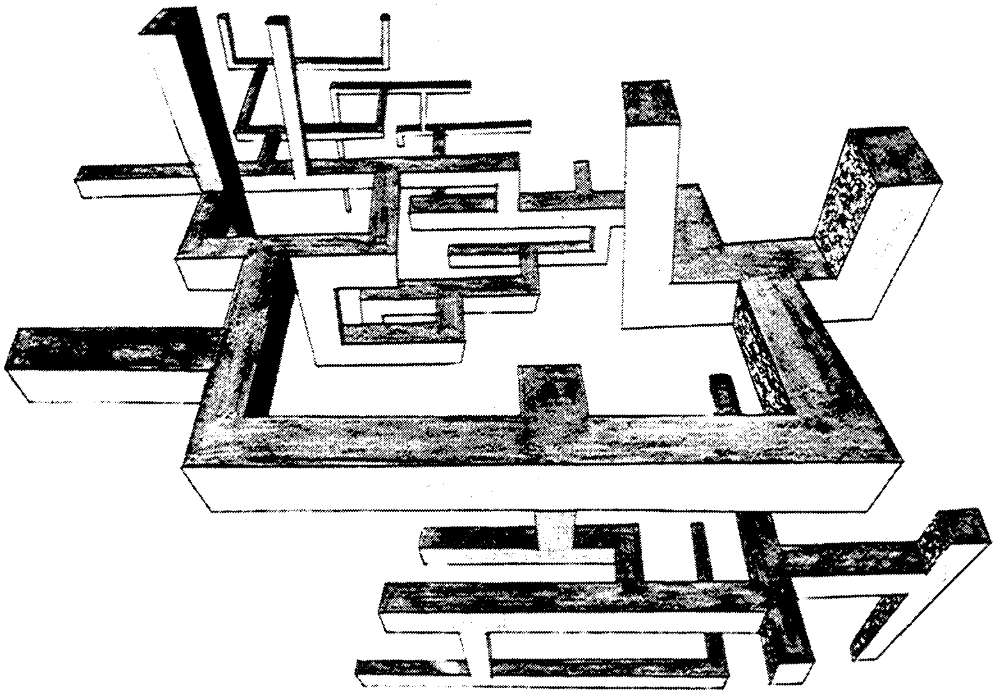


FIG. 6. After.

the amount of order in our system went up from zero to

$$R_\infty = \frac{1}{\log_2 (en)},$$

if one started out with a population density of n cubes per unit volume.

I grant you, that this increase in orderliness is not impressive at all, particularly if the population density is high. All right then, let's take a population made up entirely of members belonging to family IVB, which is characterized by opposite polarity of the two pairs of those three sides which join in two opposite corners. I put these cubes into my box and you shake it. After some time we open the box and, instead of seeing a heap of cubes piled up somewhere in the box (Fig. 5), you may not believe your eyes, but an incredibly ordered structure will emerge, which, I fancy, may pass the grade to be displayed in an exhibition of surrealistic art (Fig. 6).

If I would have left you ignorant with respect to my magnetic-surface trick and you would ask me, what is it that put these cubes into this remarkable order, I would keep a straight face and would answer: The shaking, of course—and some little demons in the box.

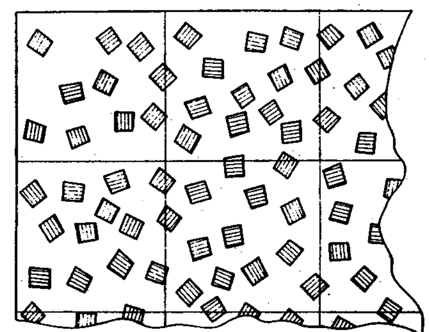
With this example, I hope, I have sufficiently illustrated the principle I called "order from noise," because no order was fed to the system, just cheap undirected energy; however, thanks to the little demons in the box, in the long run only those components of the noise were selected which contributed to the increase of order in the system. The occurrence of a mutation e.g. would be a pertinent analogy in the case of gametes being the systems of consideration.

Hence, I would name two mechanisms as important clues to the understanding of self-organizing systems, one we may call the "order from order" principle as Schrodinger suggested, and the other one the "order from noise" principle, both of which require the co-operation of our demons who are created along with the elements of our system, being manifest in some of the intrinsic structural properties of these elements.

I may be accused of having presented an almost trivial case in the attempt to develop my order from noise principle. I agree. However, I am convinced that I would maintain a much stronger position, if I would not have given away my neat little trick with the magnetized surfaces. Thus, I am very grateful to the sponsors of this conference that they invited Dr. Auerbach[6] who later in this meeting will tell us about his beautiful experiments *in vitro* of the reorganization of cells into predetermined organs after the cells have been completely separated and mixed. If Dr. Auerbach happens to know the trick by which this is accomplished, I hope he does not give it away. Because, if he would remain silent, I could recover my thesis that without having some knowledge of the mechanisms involved, my example was not too trivial after all, and self-organizing systems still remain miraculous things.

APPENDIX

The entropy of a system of given size consisting of N indistinguishable elements will be computed taking only the spatial distribution


(a)

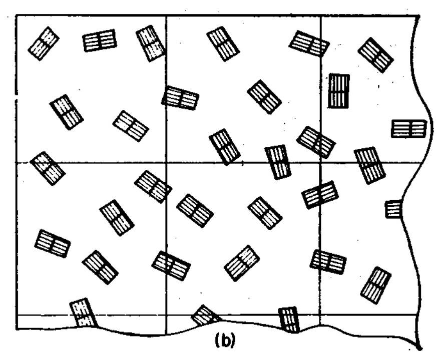

(b)

FIG. 7.

of elements into consideration. We start by subdividing the space into Z cells of equal size and count the number of cells Z_i lodging i elements (see Fig. 7*a*). Clearly we have

$$\Sigma Z_i = Z \qquad \text{(i)}$$

$$\Sigma i Z_i = N \qquad \text{(ii)}$$

The number of distinguishable variations of having a different number of elements in the cells is

$$P = \frac{Z!}{\Pi Z_i!} \qquad \text{(iii)}$$

whence we obtain the entropy of the system for a large number of cells and elements:

$$H = \ln P = Z \ln Z - \Sigma Z_i \ln Z_i \qquad \text{(iv)}$$

In the case of maximum entropy $\bar{H}$ we must have

$$\delta H = 0 \qquad \text{(v)}$$

observing also the conditions expressed in eqs. (i) and (ii). Applying the method of the Lagrange multipliers we have from (iv) and (v) with (i) and (ii):

$$\begin{array}{ll} \Sigma(\ln Z_i + 1)\delta Z_i = 0 & \\ \Sigma i \delta Z_i = 0 & \Big| \; \beta \\ \Sigma \delta Z_i = 0 & \Big| \; -(1 + \ln \alpha) \end{array}$$

multiplying with the factors indicated and summing up the three equations we note that this sum vanishes if each term vanishes identically. Hence:

$$\ln Z_i + 1 + i\beta - 1 - \ln \alpha = 0 \qquad \text{(vi)}$$

whence we obtain that distribution which maximizes H:

$$Z_i = \alpha e^{-i\beta} \qquad \text{(vii)}$$

The two undetermined multipliers α and β can be evaluated from eqs. (i) and (ii):

$$\alpha \Sigma e^{-i\beta} = Z \qquad \text{(viii)}$$

$$\alpha \Sigma i e^{-i\beta} = N \qquad \text{(ix)}$$

Remembering that

$$-\frac{\delta}{\delta\beta} \Sigma e^{-i\beta} = \Sigma i e^{-i\beta}$$

we obtain from (viii) and (ix) after some manipulation:

$$\alpha = Z(1 - e^{-1/n}) \approx \frac{Z}{n} \qquad \text{(x)}$$

$$\beta = \ln\left(1 + \frac{1}{n}\right) \approx \frac{1}{n} \qquad \text{(xi)}$$

where n, the mean cell population or density N/Z is assumed to be large in order to obtain the simple approximations. In other words, cells are assumed to be large enough to lodge plenty of elements.

Having determined the multipliers α and β, we have arrived at the most probable distribution which, after eq. (vii) now reads:

$$Z_i = \frac{Z}{n} e^{-i/n} \qquad \text{(xii)}$$

From eq. (iv) we obtain at once the maximum entropy:

$$\bar{H} = Z \ln (en). \qquad \text{(xiii)}$$

Clearly, if the elements are supposed to be able to fuse into pairs (Fig. 7*b*), we have

$$\bar{H}' = Z \ln (en/2). \qquad \text{(xiv)}$$

Equating $\bar{H}$ with H_m and $\bar{H}'$ with H, we have for the amount of order after fusion:

$$R = 1 - \frac{Z \ln (en)}{Z \ln (en/2)} = \frac{1}{\log_2 (en)} \qquad \text{(xv)}$$

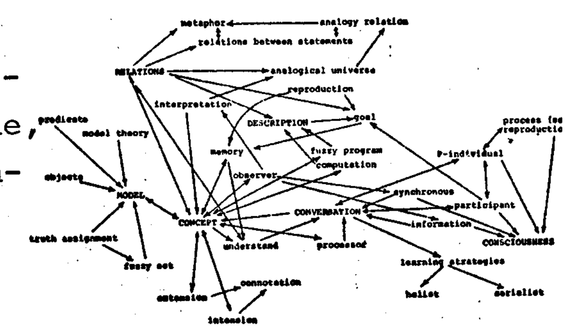
REFERENCES

1. L. Wittgenstein, *Tractatus Logico-Philosophicus*, paragraph 6.31, Humanities Publishing House, New York (1956).
2. G. A. Pask, The natural history of networks. This volume, p. 232.
3. C. Shannon and W. Weaver, *The Mathematical Theory of Communication*, p. 25, University of Illinois Press, Urbana, Illinois (1949).
4. W. Heisenberg, *Z. Phys.* **43**, 172 (1927).
5. E. Shrodinger, *What is Life?* pp. 72, 80, 80, 82, Macmillan, New York (1947).
6. R. Auerbach, Organization and reorganization of embryonic cells. This volume, p. 101.

SELF ORGANIZING SYSTEM

A non-stationary system becomes 'self-organizing' when there is uncertainty about the criteria of macroscopic similarity. An observer is impelled to change his criteria of similarity (hence, also, his reference frame) in order to make sense of the self-organizing systems' behaviour and he changes it on the basis of what he has already learned (by his interaction with the system). Typically self-organizing systems are 'alive' though we shall examine some which have been embodied in 'inanimate' materials. Let us take 'man', whom most of us would agree is a self-organizing system. A man is any member of a well-specified set of men. But this set can be well-specified (that is, specified in a way that meets common approval) in a vast number of ways, according to an observer's objective. Man, for example, may be specified anatomically (two legs, head, and so on), or alternatively as a decision maker which influences and is influenced by his circle of acquaintances. Each specification is equally valid and entails criteria of similarity. The point is, there are objectives for which neither the first specification (and the criteria it entails), nor the second (and the criteria it entails) are sufficient. In conversation, when trying to control a man, to persuade him to do something, how do I define him? Manifestly, I do not, at least, I continually change my specification in such a way that he appears <u>to me</u> as a self-organizing system.

Hence, the phrase 'self-organizing system', entails a <u>relation between</u> an observer and an assembly. It also entails the observer's objective (an assembly may be a self-organizing system for one observer but not another, or for one objective but not another). Again it is possible that an assembly will appear as a self-organizing system initially and become stationary after interaction (the conversation partner does, on average, what I ask him). The dependence is also evident in measures of organization; for example, Von Foerster proposes to use Shannon's Redundancy for this purpose. A system is 'self-organizing' if the rate of change of its redundancy is positive. [G.P.]



Ashby

W. ROSS ASHBY
University of Illinois

PRINCIPLES OF THE SELF-ORGANIZING SYSTEM*

Questions of principle are sometimes regarded as too unpractical to be important, but I suggest that that is certainly not the case in *our* subject. The range of phenomena that we have to deal with is so broad that, were it to be dealt with wholly at the technological or practical level, we would be defeated by the sheer quantity and complexity of it. The total range can be handled only piecemeal; among the pieces are those homomorphisms of the complex whole that we call "abstract theory" or "general principles". They alone give the bird's-eye view that enables us to move about in this vast field without losing our bearings. I propose, then, to attempt such a bird's-eye survey.

WHAT IS "ORGANIZATION"?

At the heart of our work lies the fundamental concept of "organization". What do we mean by it? As it is used in biology it is a somewhat complex concept, built up from several more primitive concepts. Because of this richness it is not readily defined, and it is interesting to notice that while March and Simon (1958) use the word "Organizations" as title for their book, they do not give a formal definition. Here I think they are right, for the word covers a multiplicity of meanings. I think that in future we shall hear the *word* less frequently, though the *operations* to which it corresponds, in the world of computers and brain-like mechanisms, will become of increasing daily importance.

The hard core of the concept is, in my opinion, that of "conditionality". As soon as the relation between two entities *A* and *B* becomes conditional on *C*'s value or state then a necessary component of "organization" is present. Thus *the theory of organization is partly co-extensive with the theory of functions of more than one variable.*

We can get another angle on the question by asking "what is its converse?" The converse of "conditional on" is "not conditional on", so the converse of "organization" must therefore be, as the mathematical theory shows as clearly, the concept of "reducibility". (It is also called "separability".) This occurs, in mathematical forms, when what looks like a function of several variables (perhaps very many) proves on closer examination to have parts whose actions are *not* conditional on the values of the other parts. It occurs in mechanical forms, in hardware, when what looks like one machine proves to be composed of two (or more) sub-machines, each of which is acting independently of the others.

Questions of "conditionality", and of its converse "reducibility", can, of course, be treated by a number of mathematical and logical methods. I shall say something of such methods later. Here, however, I would like to express the opinion that the method of Uncertainty Analysis, introduced by Garner and McGill (1956), gives us a method for the treatment of conditionality that is not only completely rigorous but is also of extreme generality. Its great generality and suitability for application to complex behavior, lies in the fact that it is applicable to any arbitrarily defined set of states. Its application requires neither linearity, nor continuity, nor a metric, nor even an ordering relation. By this calculus, the *degree* of conditionality can be measured, and analyzed, and apportioned to factors and interactions in a manner exactly parallel to Fisher's method of the analysis of variance; yet it requires no metric in the variables, only the frequencies with which the various combinations of states occur. It seems to me that, just as Fisher's conception of the analysis of variance threw a flood of light on to the complex relations that may exist between variations on a metric, so McGill and Garner's conception of uncertainty analysis may give us an altogether better understanding of how to treat complexities of relation when the variables are non-metric. In psychology and biology such variables occur with great commonness; doubtless they will also occur commonly in the brain-like

* The work on which this paper is based was supported by ONR Contract N 049–149.

processes developing in computers. I look forward to the time when the methods of McGill and Garner will become the accepted language in which such matters are to be thought about and treated quantitatively.

The treatment of "conditionality" (whether by functions of many variables, by correlation analysis, by uncertainty analysis, or by other ways) makes us realize that the essential idea is that there is first a product space—that of the *possibilities*—within which some sub-set of points indicates the actualities. This way of looking at "conditionality" makes us realize that it is related to that of "communication"; and it is, of course, quite plausible that we should define parts as being "organized" when "communication" (in some generalized sense) occurs between them. (Again the natural converse is that of independence, which represents non-communication.)

Now "communication" from *A* to *B* necessarily implies some constraint, some correlation between what happens at *A* and what at *B*. If, for given event at *A*, all possible events may occur at *B*, then there is no communication from *A* to *B* and no constraint over the possible (*A*, *B*)-couples that can occur. Thus the presence of "organization" between variables is equivalent to the existence of a *constraint* in the product-space of the possibilities. I stress this point because while, in the past, biologists have tended to think of organization as something extra, something *added* to the elementary variables, the modern theory, based on the logic of communication, regards organization as a restriction or constraint. The two points of view are thus diametrically opposed; there is no question of either being exclusively right, for each can be appropriate in its context. But with this opposition in existence we must clearly go carefully, especially when we discuss with others, lest we should fall into complete confusion.

This excursion may seem somewhat complex but it is, I am sure, advisable, for we have to recognize that the discussion of organization theory has a peculiarity not found in the more objective sciences of physics and chemistry. The peculiarity comes in with the product space that I have just referred to. Whence comes this product space? Its chief peculiarity is that *it contains more than actually exists in the real physical world*, for it is the latter that gives us the actual, constrained *subset*.

The real world gives the subset of what *is*; the product space represents the uncertainty of the *observer*. The product space may therefore change if the observer changes; and two observers may legitimately use different product spaces within which to record the same subset of actual events in some actual thing. The "constraint" is thus a *relation* between observer and thing; the properties of any particular constraint will depend on both the real thing and on *the observer*. It follows that a substantial part of the theory of organization will be concerned with *properties that are not intrinsic to the thing but are relational between observer and thing*. We shall see some striking examples of this fact later.

WHOLE AND PARTS

"If conditionality" is an essential component in the concept of organization, so also is the assumption that we are speaking of a whole composed of parts. This assumption is worth a moment's scrutiny, for research is developing a theory of dynamics that does *not* observe parts and their interactions, but treats the system as an unanalysed whole (Ashby, 1958, a). In physics, of course, we usually start the description of a system by saying "Let the variables be $x_1, x_2, \ldots, x_n$" and thus start by treating the whole as made of *n* functional parts. The other method, however, deals with unanalysed states, $S_1, S_2, \ldots$ of the whole, without explicit mention of any parts that may be contributing to these states. The dynamics of such a system can then be defined and handled mathematically; I have shown elsewhere (Ashby, 1960, a) how such an approach can be useful. What I wish to point out here is that we can have a sophisticated *dynamics*, of a whole as complex and cross-connected as you please, that makes no reference to any parts and that therefore does *not* use the concept of organization. Thus the concepts of dynamics and of organization are essentially independent, in that all four combinations, of their presence and absence, are possible.

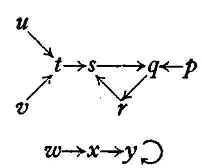
This fact exemplifies what I said, that "organization" is partly in the eye of the beholder. Two observers studying the same real material system, a hive of bees say, may find that one of them, thinking of the hive as an interaction of fifty thousand bee-parts, finds the bees "organized", while the other, observing whole states such as activity, dormancy, swarming, etc., may see *no* organization, only trajectories of these (unanalysed) states.

Another example of the independence of "organization" and "dynamics" is given by the fact that whether or not a real system is organized or reducible depends partly on the point of view taken by the observer. It is well known, for instance, that an organized (i.e. interacting) linear system of *n* parts, such as a network of pendulums and springs, can be seen from another point of view (that of the so-called "normal" coordinates) in which all the (newly identified) parts are completely separate, so that the whole is reducible. There is therefore nothing perverse about my insistence on the relativity of organization, for advantage of the fact is routinely taken in the study of quite ordinary dynamic systems.

Finally, in order to emphasize how dependent is the organization seen in a system on the observer who sees it, I will state the proposition that: given a whole with arbitrarily given behavior, a great variety of arbitrary "parts" can be seen in it; for all that is necessary, when the arbitrary part is proposed, is that we assume the given part to be coupled to another suitably related part, so that the two together form a whole isomorphic with the whole that was given. For instance, suppose the given whole, W of 10 states, behaves in accordance with the transformation:

$$W \downarrow \begin{matrix} p & q & r & s & t & u & v & w & x & y \\ q & r & s & q & s & t & t & x & y & y \end{matrix}$$

Its kinematic graph is



and suppose we wish to "see" it as containing the part P, with internal states E and input states A:

↓		E: 1	2
A	1	2	1
	2	1	1

} P

With a little ingenuity we find that if part P is coupled to part Q (with states (F, G) and input B) with transformation Q:

↓		(F, G): 1,1	1,2	1,3	2,1	2,2	2,3
B	1	2,1	1,2	1,2	2,1	1,2	1,2
	2	•	2,3	•	2,1	2,2	2,2

} Q

by putting $A = F$ and $B = E$, then the new whole W' has transformation

$$W': \downarrow \begin{matrix} 1,1,1 & 1,1,2 & 1,1,3 & 1,2,1, & \text{etc.} \\ 2,2,1 & 2,1,2 & 2,1,2 & 1,2,1, & \text{etc.} \end{matrix}$$

which is *isomorphic with W* under the one–one correspondence

$$\downarrow \begin{matrix} 1,1,1 & 1,1,2 & 1,1,3 & 1,2,1, & \text{etc.} \\ w & s & p & y\ , & \text{etc.} \end{matrix}$$

Thus, subject only to certain requirements (e.g. that equilibria map into equilibria) *any dynamic system can be made to display a variety of arbitrarily assigned "parts"*, simply by a change in the *observer's* view point.

MACHINES IN GENERAL

I have just used a way of representing two "parts", "coupled" to form a "whole", that anticipates the question: what do we mean by a "machine" in general?

Here we are obviously encroaching on what has been called "general system theory", but this last discipline always seemed to me to be uncertain whether it was dealing with *physical* systems, and therefore tied to whatever the real world provides, or with mathematical systems, in which the sole demand is that the work shall be free from internal contradictions. It is, I think, one of the substantial advances of the last decade that we have at last identified the *essentials* of the "machine in general".

Before the essentials could be seen, we had to realize that two factors must be *excluded as irrelevant*. The first is "materiality"—the idea that a machine must be made of actual matter, of the hundred or so existent elements. This is wrong, for examples can readily be given (e.g. Ashby, 1958, a) showing that what is essential is whether the system, of angels and ectoplasm if you please, *behaves* in a law-abiding and machine-like way. Also to be excluded as irrelevant is any reference to energy, for any calculating machine shows that what matters is the *regularity* of the behavior—whether energy is gained or lost, or even created, is simply irrelevant.

The fundamental concept of "machine" proves to have a form that was formulated at least a century ago, but this concept has not, so far as I am aware, ever been used and exploited vigorously. A "machine" is that which behaves in a machine-like way, namely, that its internal state, and the state of its surroundings, defines uniquely the next state it will go to.

This definition, formally proposed fifteen years ago (Ashby, 1945) has withstood the passage of time and is now becoming generally accepted (e.g. Jeffrey, 1959). It appears in many forms. When the variables are continuous it corresponds to the description of a dynamic system by giving a set of ordinary differential equations with time as the independent variable. The *fundamental* nature of such a representation (as contrasted with a merely convenient one) has been recognized by many earlier workers such as Poincaré, Lotka (1925), and von Bertalanffy (1950 and earlier).

Such a representation by differential equations is, however, too restricted for the needs of a science that includes biological systems and calculating machines, in which discontinuity is ubiquitous. So arises the modern definition, able to include both the continuous and the discontinuous and even the discrete, without the slightest loss of rigor. The "machine with input" (Ashby, 1958, a) or the "finite automaton" (Jeffrey, 1959) is today defined by a set S of internal states, a set I of input or surrounding states, and a mapping, f say, of the product set $I \times S$ into S. Here, in my opinion, we have the very essence of the "machine"; all known types of machine are to be found here; and all interesting deviations from the concept are to be found by the corresponding deviation from the definition.

We are now in a position to say without ambiguity or evasion what we mean by a machine's "organization". First we specify which system we are talking about by specifying its states S and its conditions I. If S is a product set, so that $S = \Pi_i T_i$ say, then the parts i are each specified by its set of states T_i. *The "organization" between these parts is then specified by the mapping f.* Change f and the organization changes. In other words, the possible organizations between the parts can be set into one–one correspondence with the set of possible mappings of $I \times S$ into S. Thus "organization" and "mapping" are two ways of looking at the same thing—the organization being noticed by the observer of the actual system, and the mapping being recorded by the person who represents the behavior in mathematical or other symbolism.

"GOOD" ORGANIZATION

At this point some of you, especially the biologists, may be feeling uneasy; for this definition of organization makes no reference to any *usefulness* of the organization. It demands only that there be conditionality between the parts and regularity in behavior. In this I believe the definition to be right, for the question whether a given organization is "good" or "bad" is quite independent of the prior test of whether it is or is not an organization.

I feel inclined to stress this point, for here the engineers and the biologists are likely to think along widely differing lines. The engineer, having put together some electronic hardware and having found the assembled network to be roaring with parasitic oscillations, is quite accustomed to the idea of a "bad" organization; and he knows that the "good" organization has to be searched for. The biologist, however, studies mostly animal species that have survived the long process of natural selection; so almost all the organizations he sees have already been selected to be good ones, and he is apt to think of "organizations" as *necessarily* good. This point of view may often be true in the biological world but it is most emphatically not true in the world in which we people here are working. We *must* accept that

(1) most organizations are bad ones;
(2) the good ones have to be sought for; and
(3) what is meant by "good" must be clearly defined, explicitly if necessary, *in every case*.

What then is meant by "good", in our context of brain-like mechanisms and computers? We must proceed cautiously, for the word suggests some evaluation whose origin has not yet been considered.

In some cases the distinction between the "good" organization and the "bad" is obvious, in the sense that as everyone in these cases would tend to use the same criterion, it would not need explicit mention. The brain of a living organism, for instance, is usually judged as having a "good" organization if the organization (whether inborn or learned) acts so as to further the organism's survival. This consideration readily generalizes to all those cases in which the organization (whether of a cat or an automatic pilot or an oil refinery) is judged "good" if and only if it acts so as to keep an assigned set of variables, the "essential" variables, within assigned limits. Here are all the mechanisms for homeostasis, both in the original sense of Cannon and in the generalized sense. From this criterion comes the related one that an organization is "good" if it makes the system stable around an assigned equilibrium. Sommerhoff (1950) in particular has given a wealth of examples, drawn from a great range of biological and mechanical phenomena, showing how in all cases the idea of a "good organization" has as its essence the idea of a number of parts so interacting as to achieve some given "focal condition". I would like to say here that I do not consider that Sommerhoff's contribution to our subject has yet been adequately recognized. His identification of *exactly* what is meant by coordination and integration is, in my opinion, on a par with Cauchy's identification of exactly what was meant by convergence. Cauchy's discovery was a real discovery, and was an enormous help to later workers by providing them with a concept, rigorously defined, that could be used again and again, in a vast range of contexts, and always with exactly the same meaning. Sommerhoff's discovery of how to represent *exactly* what is meant by coordination and integration and good organization will, I am sure, eventually play a similarly fundamental part in our work.

His work illustrates, and emphasizes, what I want to say here—*there is no such thing as "good organization" in any absolute sense.* Always it is relative; and an organization that is good in one context or under one criterion may be bad under another.

Sometimes this statement is so obvious as to arouse no opposition. If we have half a dozen lenses, for instance, that can be assembled this way to make a telescope or that way to make a microscope, the goodness of an assembly obviously depends on whether one wants to look at the moon or a cheese mite.

But the subject is more contentious than that! The thesis implies that there is no such thing as a brain (natural or artificial) that is good in any absolute sense—it all depends on the circumstances and on what is wanted. Every faculty that a brain can show is "good" only conditionally, for there exists at least one environment against which the brain is handicapped by the possession of this faculty. Sommerhoff's formulation enables us to show this at once: whatever the faculty or organization achieves, let that be *not* in the "focal conditions".

We know, of course, lots of examples where the thesis is true in a somewhat trivial way. Curiosity tends to be good, but many an antelope has lost its life by stopping to see what the hunter's hat is. Whether the organization of the antelope's brain should be of the type that does, or does not, lead to temporary immobility clearly depends on whether hunters with rifles are or are not plentiful in its world.

From a different angle we can notice Pribram's results (1957), who found that brain-operated monkeys scored higher in a certain test than the normals. (The operated were plodding and patient while the normals were restless and distractible.) Be that as it may, one cannot say which brain (normal or operated) had the "good" organization until one has decided which sort of temperament is wanted.

Do you still find this non-contentious? Then I am prepared to assert that there is not a single mental faculty ascribed to Man that is good in the absolute sense. If any particular faculty is *usually* good, this is solely because our terrestrial environment is so lacking in variety that its usual form makes that faculty usually good. But change the environment, go to really different conditions, and possession of that faculty may be harmful. And "bad", by implication, is the brain organization that produces it.

I believe that there is not a single faculty or property of the brain, usually regarded as desirable, that does not become *un*desirable in some type of environment. Here are some examples in illustration.

The first is Memory. Is it not good that a brain should have memory? Not at all, I reply—only when the environment is of a type in which the future often *copies* the past; should the future often be the *inverse* of the past, memory is actually disadvantageous. A well known example is given when the sewer rat faces the environmental system known as "pre-baiting". The naïve rat is very suspicious, and takes strange food only in small quantities. If, however, wholesome food appears at some place for three days in succession, the sewer rat will learn, and on the fourth day will eat to repletion, and die. The rat without memory, however, is as suspicious on the fourth day as on the first, and lives. Thus, in *this* environment, memory is positively disadvantageous. Prolonged contact with this environment will lead, other things being equal, to evolution in the direction of diminished memory-capacity.

As a second example, consider organization itself in the sense of connectedness. Is it not good that a brain should have its parts in rich functional connection? I say, No—not *in general*; only when the environment is itself richly connected. When the environment's parts are *not* richly connected (when it is highly reducible, in other words), adaptation will go on faster if the brain is also highly reducible, i.e. if its connectivity is small (Ashby, 1960, d). Thus the *degree* of organization can be too high as well as too low; the degree we humans possess is probably adjusted to be somewhere near the optimum for the usual terrestrial environment. It does not in any way follow that this degree will be optimal or good if the brain is a mechanical one, working against some grossly non-terrestrial environment—one existing only inside a big computer, say.

As another example, what of the "organization" that the biologist always points to with pride—the development in evolution of specialized organs such as brain, intestines, heart and blood vessels. Is not this good? Good or not, it is certainly a specialization made possible only because the earth has an atmosphere; without it, we would be incessantly bombarded by tiny meteorites, any one of which, passing through our chest, might strike a large blood vessel and kill us. Under such conditions a better form for survival would be the slime mould, which specializes in being able to flow through a tangle of twigs without loss of function. Thus the development of organs is not good unconditionally, but is a specialization to a world free from flying particles.

After these actual instances, we can return to theory. It is here that Sommerhoff's formulation gives such helpful clarification. He shows that in all cases there must be given, and specified, first a *set of disturbances* (values of his "coenetic variable") and secondly a goal (his "focal condition"); the disturbances threaten to drive the outcome outside the focal condition. The "good" organization is then of the nature of a *relation* between the set of disturbances and the goal. Change the set of disturbances, and the organization, without itself changing, is evaluated "bad" instead of "good". As I said, there is no property of an organization that is good in any absolute sense; all are relative to some given environment, or to some given set of threats and disturbances, or to some given set of problems.

SELF-ORGANIZING SYSTEMS

I hope I have not wearied you by belaboring this relativity too much, but it is fundamental, and is only too readily forgotten when one comes to deal with organizations that are either biological in origin or are in imitation of such systems. With this in mind, we can now start to consider the so-called "self-organizing" system. We must proceed with some caution here if we are not to land in confusion, for the adjective is, if used loosely, ambiguous, and, if used precisely, self-contradictory.

To say a system is "self-organizing" leaves open two quite different meanings.

There is a first meaning that is simple and unobjectionable. This refers to the system that starts with its parts separate (so that the behavior of each is independent of the others' states) and whose parts then act so that they change towards forming connections of some type. Such a system is "self-organizing" in the sense that it changes from "parts separated" to "parts joined". An example is the embryo nervous system, which starts with cells having little or no effect on one another, and changes, by the growth of dendrites and formation of synapses, to one in which each part's behavior is very much affected by the other parts. Another example is Pask's system of electrolytic centers, in which the growth of a filament from one electrode is at first little affected by growths at the other electrodes; then the growths become more and more affected by one another as filaments approach the other electrodes. In general such systems can be more simply characterized as "self-*connecting*", for the change from independence between the parts to conditionality can always be seen as some form of "connection", even if it is as purely functional as that from a radio transmitter to a receiver.

Here, then, is a perfectly straightforward form of self-organizing system; but I must emphasize that there can be no assumption at this point that the organization developed will be a good one. If we wish it to be a "good" one, we must first provide a criterion for distinguishing between the bad and the good, and then we must ensure that the appropriate selection is made.

We are here approaching the second meaning of "self-organizing" (Ashby, 1947). "Organizing" may have the first meaning, just discussed, of "changing from unorganized to organized". But it may also mean "changing from a bad organization to a good one", and this is the case I wish to discuss now, and more fully. This is the case of peculiar interest to *us*, for this is the case of the system that changes itself from a bad way of behaving to a good. A well known example is the child that starts with a brain organization that makes it fire-seeking; then a change occurs, and a new brain organization appears that makes the child fire-avoiding. Another example would occur if an automatic pilot and a plane were so coupled, by mistake, that positive feedback made the whole error-aggravating rather than error-correcting. Here the organization is bad. The system would be "self-organizing" if a change were *automatically* made to the feedback, changing it from positive to negative; then the whole would have changed from a bad organization to a good. Clearly, *this* type of "self-organization" is of peculiar interest to us. What is implied by it?

Before the question is answered we must notice, if we are not to be in perpetual danger of confusion, that *no machine can be self-organizing in this sense*. The reasoning is simple. Define the set S of states so as to specify which machine we are talking about. The "organization" must then, as I said above, be identified with f, the mapping of S into S that the basic drive of the machine (whatever force it may be) imposes. Now the logical relation here is that f determines the changes of S:—f is *defined* as the set of couples (s_i, s_j) such that the internal drive of the system will force state s_i to change to s_j. To allow f to be a function of the state is to make nonsense of the whole concept.

Since the argument is fundamental in the theory of self-organizing systems, I may help explanation by a parallel example. Newton's law of gravitation says that $F = M_1M_2/d^2$, in particular, that the force varies inversely as the distance to power 2. To power 3 would be a different law. But suppose it were suggested that, not the force F but the *law* changed with the distance, so that the power was not 2 but some function of the distance, $\phi(d)$. This suggestion is illogical; for we now have that $F = M_1M_2/d^{\phi(d)}$, and this represents not a law that varies with the distance but *one* law covering all distances; that is, were this the case we would *re-define* the law. Analogously, were f in the machine to be some function of the state S, we would have to re-define our machine. Let me be quite explicit with an example. Suppose S had three states: a, b, c. If f depended on S there would be three f's: f_a, f_b, f_c say. Then if they are

↓	a	b	c
f_a	**b**	a	b
f_b	c	**a**	a
f_c	b	b	**a**

then the transform of a must be under f_a, and is therefore b, so the whole set of f's would amount to the *single* transformation:

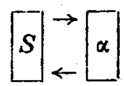
$$\downarrow \begin{matrix} a & b & c \\ b & a & a \end{matrix}$$

It is clearly illogical to talk of f as being a function of S, for such talk would refer to operations, such as $f_a(b)$, which cannot in fact occur.

If, then, no machine can properly be said to be self-organizing, how do we regard, say, the Homeostat, that rearranges its own wiring; or the computer that writes out its own program?

The new logic of mechanism enables us to treat the question rigorously. We start with the set S of states, and assume that f changes, to g say. So we really have a *variable*, $\alpha(t)$ say, a function of the time that had at first the value f and later the value g. This change, as we have just seen, cannot be ascribed to any cause in the set S; so it must have come from some outside agent, acting on the system S as input. If the system is to be in some sense "*self*-organizing", the "self" must be enlarged to include this variable α, and, to keep the whole bounded, the cause of α's change must be in S (or α).

Thus the appearance of being "self-organizing" can be given only by the machine S being coupled to another machine (of one part):



Then the part S can be "self-organizing" within the whole $S+\alpha$.

Only in this partial and strictly qualified sense can we understand that a system is "*self*-organizing" without being self-contradictory.

Since no system can correctly be said to be self-organizing, and since use of the phrase "self-organizing" tends to perpetuate a fundamentally confused and inconsistent way of looking at the subject, the phrase is probably better allowed to die out.

THE SPONTANEOUS GENERATION OF ORGANIZATION

When I say that no system can properly be said to be self-organizing, the listener may not be satisfied. What, he may ask, of those changes that occurred a billion years ago, that led lots of carbon atoms, scattered in little molecules of carbon dioxide, methane, carbonate, etc., to get together until they formed proteins, and then went on to form those large active lumps that today we call "animals"? Was not this process, on an isolated planet, one of "self-organization"? And if it occurred on a planetary surface can it not be made to occur in a computer? I am, of course, now discussing the origin of life. Has modern system theory anything to say on this topic?

It has a great deal to say, and some of it flatly contradictory to what has been said ever since the idea of evolution was first considered. In the past, when a writer discussed the topic, he usually assumed that the generation of life was rare and peculiar, and he then tried to display some way that would enable this rare and peculiar event to occur. So he tried to display that there is *some* route from, say, carbon dioxide to the amino acid, and thence to the protein, and so, through natural selection and evolution, to intelligent beings. I say that this looking for special conditions is quite wrong. The truth is the opposite—*every* dynamic system generates its own form of intelligent life, is self-organizing in this sense. (I will demonstrate the fact in a moment.) Why we have failed to recognize this fact is that until recently we have had no experience of systems of medium complexity; either they have been like the watch and the pendulum, and we have found their properties few and trivial, or they have been like the dog and the human being, and we have found their properties so rich and remarkable that we have thought them supernatural. Only in the last few years has the general-purpose computer given us a system rich enough to be interesting yet still simple enough to be understandable. With this machine as tutor we can now begin to think about systems that are simple enough to be comprehensible in detail yet also rich enough to be suggestive. With their aid we can see the truth of the statement that *every isolated determinate dynamic system obeying unchanging laws will develop "organisms" that are adapted to their "environments"*.

The argument is simple enough in principle. We start with the fact that systems in general go to equilibrium. Now most of a system's states are non-equilibrial (if we exclude the extreme case of the system in neutral equilibrium). So in going from *any* state to one of the equilibria, the system is going from a larger number of states to a smaller. In this way it is performing a selection, in the purely objective sense that it rejects some states, by leaving them, and retains some other state, by sticking to it. Thus, as every determinate system goes to equilibrium, so does it select. We have heard *ad nauseam* the dictum that a machine cannot select; the truth is just the opposite: every machine, as it goes to equilibrium, performs the corresponding act of selection.

Now, equilibrium in simple systems is usually trivial and uninteresting; it is the pendulum hanging vertically; it is the watch with its main-spring run down; the cube resting flat on one face. Today, however, we know that when the system is more complex and dynamic, equilibrium, and the stability around it, can be much more interesting. Here we have the automatic pilot successfully combating an eddy; the person redistributing his blood flow after a severe haemorrhage; the business firm restocking after a sudden increase in consumption; the economic system restoring a distribution of supplies after a sudden destruction of a food crop; and it is a man successfully getting at least one meal a day during a lifetime of hardship and unemployment.

What makes the change, from trivial to interesting, is simply the *scale* of the events. "Going to equilibrium" *is* trivial in the simple pendulum, for the equilibrium is no more than a single point. But when the system is more complex; when, say, a country's economy goes back from wartime to normal methods then the stable region is vast, and much interesting activity can occur within it. The computer is heaven-sent in this context, for it enables us to bridge the enormous conceptual gap from the simple and understandable to the complex and interesting. Thus we can gain a considerable insight into the so-called spontaneous generation of life by just seeing how a somewhat simpler version will appear in a computer.

COMPETITION

Here is an example of a simpler version. The competition between species is often treated as if it were essentially biological; it is in fact an expression of a process of far greater generality. Suppose we have a computer, for instance, whose stores are filled at random with the digits 0 to 9. Suppose its dynamic law is that the digits are continuously being multiplied in pairs, and the right-hand digit of the product going to replace the first digit taken. Start the machine, and let it "evolve"; what will happen? Now under the laws of this particular world, even times even gives even, and odd times odd gives odd. But even times odd gives even; so after a mixed encounter *the even has the better chance of survival.* So as this system evolves, we shall see the evens favored in the struggle, steadily replacing the odds in the stores and eventually exterminating them.

But the evens are not homogeneous, and among them the zeros are best suited to survive in this particular world; and, as we watch, we shall see the zeros exterminating their fellow-evens, until eventually they inherit this particular earth.

What we have here is an example of a thesis of extreme generality. From one point of view we have simply a well defined operator (the multiplication and replacement law) which drives on towards equilibrium. In doing so it *automatically* selects those operands that are *specially resistant* to its change-making tendency (for the zeros are uniquely resistant to change by multiplication). This process, of progression towards the specially resistant form, is of extreme generality, demanding only that the operator (or the physical laws of any physical system) be determinate and unchanging. This is the general or abstract point of view. The biologist sees a special case of it when he observes the march of evolution, survival of the fittest, and the inevitable emergence of the highest biological functions and intelligence. Thus, when we ask: What was necessary that life and intelligence should appear? the answer is not carbon, or amino acids or any other special feature but only that the dynamic laws of the process should be *unchanging*, i.e. that the system should be *isolated. In any isolated system, life and intelligence inevitably develop* (they may, in degenerate cases, develop to only zero degree).

So the answer to the question: How can we generate intelligence synthetically? is as follows. Take a dynamic system whose laws are unchanging and single-valued, and whose size is so large that after it has gone to an equilibrium that involves only a small fraction of its total states, this small fraction is still large enough to allow room for a good deal of change and behavior. Let it go on for a long enough time to get to such an equilibrium. Then examine the equilibrium in detail. You will find that the states or forms now in being are peculiarly able to survive against the changes induced by the laws. Split the equilibrium in two, call one part "organism" and the other part "environment": you will find that this "organism" is peculiarly able to survive against the disturbances from this "environment". The *degree* of adaptation and complexity that this organism can develop is bounded only by the size of the whole dynamic system and by the time over which it is allowed to progress towards equilibrium. Thus, as I said, every isolated determinate dynamic system will develop organisms that are adapted to their environments. There is thus no difficulty in principle, in developing synthetic organisms as complex or as intelligent as we please.

In *this* sense, then, *every* machine can be thought of as "self-organizing", for it will develop, to such degree as its size and complexity allow, some functional structure homologous with an "adapted organism". But does this give us what we at this Conference are looking for? Only partly; for nothing said so far has any implication about the organization being good or bad; the criterion that would make the distinction has not yet been introduced. It is true, of course, that the developed organism, being stable, will have its own essential variables, and it will show its stability by vigorous reactions that tend to preserve its own existence. To *itself*, its own organization will *always*, by definition, be good. The wasp finds the stinging reflex a good thing, and the leech finds the blood-sucking reflex a good thing. But these criteria come *after* the organization for survival; having seen *what* survives we then see what is "good" for that form. What emerges depends simply on what are the system's laws and from what state it started; there is no implication that the organization developed will be "good" in any absolute sense, or according to the criterion of any outside body such as ourselves.

To summarize briefly: there is no difficulty, in principle, in developing *synthetic organisms as complex, and as intelligent as we please*. But we must notice two fundamental qualifications; first, their intelligence will be an adaptation to, and a specialization towards, their particular environment, with no implication of validity for any other environment such as ours; and secondly, their intelligence will be directed towards keeping their own essential variables within limits. They will be fundamentally selfish. So we now have to ask: In view of these qualifications, can we yet turn these processes to our advantage?

REQUISITE VARIETY

In this matter I do not think enough attention has yet been paid to Shannon's Tenth Theorem (1949) or to the simpler "law of requisite variety" in which I have expressed the same basic idea (Ashby, 1958, a). Shannon's theorem says that if a correction-channel has capacity H, then equivocation of amount H can be removed, *but no more*. Shannon stated his theorem in the context of telephone or similar communication, but the formulation is just as true of a biological regulatory channel trying to exert some sort of corrective control. He thought of the case with a lot of message and a little error; the biologist faces the case where the "message" is small but the disturbing errors are many and large. The theorem can then be applied to the brain (or any other regulatory and selective device), when it says that the amount of regulatory or selective action that the brain can achieve is absolutely bounded by its capacity as a channel (Ashby, 1958, b). Another way of expressing the same idea is to say that any quantity K of appropriate selection demands the transmission or processing of quantity K of information (Ashby, 1960, b.) *There is no getting of selection for nothing*.

I think that here we have a principle that we shall hear much of in the future, for it dominates all work with complex systems. It enters the subject somewhat as the law of conservation of energy enters power engineering. When that law first came in, about a hundred years ago, many engineers thought of it as a disappointment, for it stopped all hopes of perpetual motion. Nevertheless, it did in fact lead to the great practical engineering triumphs of the nineteenth century, because it made power engineering more realistic.

I suggest that when the full implications of Shannon's Tenth Theorem are grasped we shall be, first sobered, and then helped, for we shall then be able to focus our activities on the problems that are properly realistic, and actually solvable.

THE FUTURE

Here I have completed this bird's-eye survey of the principles that govern the self-organizing system. I hope I have given justification for my belief that these principles, based on the logic of mechanism and on information theory, are now essentially *complete*, in the sense that there is now no area that is grossly mysterious.

Before I end, however, I would like to indicate, very briefly, the directions in which future research seems to me to be most likely to be profitable.

One direction in which I believe a great deal to be readily discoverable, is in the discovery of new types of dynamic process. Most of the machine-processes that we know today are very specialized, depending on exactly what parts are used and how they are joined together. But there are systems of more net-like construction in which what happens can only be treated statistically. There are processes here like, for instance, the spread of epidemics, the fluctuations of animal populations over a territory, the spread of wave-like phenomena over a nerve-net. These processes are, in themselves, neither good nor bad, but they exist, with all their curious properties, and doubtless the brain will use them should they be of advantage. What I want to emphasize here is that they often show very surprising and peculiar properties; such as the tendency, in epidemics, for the outbreaks to occur in waves. Such peculiar new properties may be just what some machine designer wants, and that he might otherwise not know how to achieve.

The study of such systems must be essentially statistical, but this does not mean that each system must be individually stochastic. On the contrary, it has recently been shown (Ashby, 1960, c) that no system can have greater efficiency than the determinate when acting as a regulator; so, as regulation is the one function that counts biologically, we can expect that natural selection will have made the brain as determinate as possible. It follows that we can confine our interest to the lesser range in which the sample space is over a set of mechanisms each of which is individually determinate.

As a particular case, a type of system that deserves much more thorough investigation is the large system that is built of parts that have many states of equilibrium. Such systems are extremely common in the terrestrial world; they exist all around us, and in fact, intelligence as we know it would be almost impossible otherwise (Ashby, 1960, d). This is another way of referring to the system whose variables behave largely as part-functions. I have shown elsewhere (Ashby, 1960, a) that such systems tend to show habituation (extinction) and to be able to adapt progressively (Ashby, 1960, d). There is reason to believe that some of the well-known but obscure biological phenomena such as conditioning, association, and Jennings' (1906) law of the resolution of physiological states may be more or less simple and direct expressions of the multiplicity of equilibrial states. At the moment I am investigating the possibility that the transfer of "structure", such as that of three-dimensional space, into a dynamic system—the sort of learning that Piaget has specially considered—may be an *automatic* process when the input comes to a system with many equilibria. Be that as it may, there can be little doubt that the study of such systems is likely to reveal a variety of new dynamic processes, giving us dynamic resources not at present available.

A particular type of system with many equilibria is the system whose parts have a high "threshold"—those that tend to stay at some "basic" state unless some function of the input exceeds some value. The general properties of such systems is still largely unknown, although Beurle (1956) has made a most interesting start. They deserve extensive investigation; for, with their basic tendency to develop avalanche-like waves of activity, their dynamic properties are likely to prove exciting and even dramatic. The fact that the mammalian brain uses the property extensively suggests that it may have some peculiar, and useful, property not readily obtainable in any other way.

Reference to the system with many equilibria brings me to the second line of investigation that seems to me to be in the highest degree promising—I refer to the discovery of *the living organism's memory store*: the identification of its physical nature.

At the moment, our knowledge of the living brain is grossly out of balance. With regard to what happens from one millisecond to the next we know a great deal, and many laboratories are working to add yet more detail. But when we ask what happens in the brain from one hour to the next, or from one year to the next, practically nothing is known. Yet it is these longer-term changes that are the really significant ones in human behavior.

It seems to me, therefore, that if there is one thing that is crying out to be investigated it is the physical basis of the brain's memory-stores. There was a time when "memory" was a very vague and metaphysical subject; but those days are gone. "Memory", as a *constraint* holding over events of the past and the present, and a *relation* between them, is today firmly grasped by the logic of mechanism. We know exactly what we mean by it behavioristically and operationally. What we need now is the provision of adequate resources for its investigation. Surely the time has come for the world to be able to find resources for *one* team to go into the matter?

SUMMARY

Today, the principles of the self-organizing system are known with some completeness, in the sense that no major part of the subject is wholly mysterious.

We have a secure base. Today we know *exactly* what we mean by "machine", by "organization", by "integration", and by "self-organization". We understand these concepts as thoroughly and as rigorously as the mathematician understands "continuity" or "convergence".

In these terms we can see today that the artificial generation of dynamic systems with "life" and "intelligence" is not merely simple—it is unavoidable if only the basic requirements are met. These are not carbon, water, or any other material entities but the persistence, over a long time, of the action of any operator that is both unchanging and single-valued. *Every* such operator forces the development of its own form of life and intelligence.

But will the forms developed be of use to *us*? Here the situation is dominated by the basic law of requisite variety (and Shannon's Tenth Theorem), which says that the achieving of appropriate selection (to a degree better than chance) is absolutely dependent on the processing of at least that quantity of information. Future work must respect this law, or be marked as futile even before it has started.

Finally, I commend as a program for research, the *identification of the physical basis of the brain's memory stores.* Our knowledge of the brain's functioning is today grossly out of balance. A vast amount is known about how the brain goes from state to state at about millisecond intervals; but when we consider our knowledge of the basis of the important long-term changes we find it to amount, practically, to nothing. I suggest it is time that we made some definite attempt to attack this problem. Surely it is time that the world had *one* team active in this direction?

REFERENCES

1. W. Ross Ashby, The physical origin of adaptation by trial and error, *J. Gen. Psychol.* **32**, pp. 13–25 (1945).
2. W. Ross Ashby, Principles of the self-organizing dynamic system. *J. Gen. Psychol.* **37**, pp. 125–8 (1947).
3. W. Ross Ashby, *An Introduction to Cybernetics*, Wiley, New York, 3rd imp. (1958, a).
4. W. Ross Ashby, Requisite variety and its implications for the control of complex systems, *Cybernetica*, **1**, pp. 83–99 (1958, b).
5. W. Ross Ashby, The mechanism of habituation. In: *The Mechanization of thought Processes.* (Natl. Phys. Lab. Symposium No. 10) H.M.S.O., London (1960).
6. W. Ross Ashby, Computers and decision-making, *New Scientist*, **7**, p. 746 1960, b).
7. W. Ross Ashby, The brain as regulator, *Nature, Lond.* **186**, p. 413 (1960, c).
8. W. Ross Ashby, *Design for a Brain; the Origin of Adaptive Behavior*, Wiley, New York, 2nd ed. (1960, d).
9. L. von Bertalanffy, An outline of general system theory, *Brit. J. Phil. Sci.* **1**, pp. 134–65 (1950).
10. R. L. Beurle, Properties of a mass of cells capable of regenerating pulses, *Proc. Roy. Soc.* **B240**, pp. 55–94 (1956).
11. W. R. Garner and W. J. McGill, The relation between information and variance analyses, *Psychometrika* **21**, pp. 219–28 (1956).
12. R. C. Jeffrey, Some recent simplifications of the theory of finite automata. Technical Report 219, Research Laboratory of Electronics, Massachusetts Institute of Technology (27 May 1959).
13. H. S. Jennings, *Behavior of the Lower Organisms*, New York (1906).
14. A. J. Lotka, *Elements of Physical Biology*, Williams & Wilkins, Baltimore (1925).
15. J. G. March and J. A. Simon, *Organizations*, Wiley, New York (1958).
16. K. H. Pribram, Fifteenth International Congress of Psychology, Brussels (1957).
17. C. E. Shannon and W. Weaver, *The Mathematical Theory of Communication*, University of Illinois Press, Urbana (1949).
18. G. Sommerhoff, *Analytical Biology*, Oxford University Press, London (1950).